Dynamics of A Degenerate Fokker-Planck Equation and Its Application

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Abstract

In this project, a Fokker-Planck equation with two singular points is studied. The equation is derived from a stochastic evolution equation, LMM-SABR model, which is widely used in financial industry.

It is difficult to directly study the original equation due to the singularity. As an alternative approach, we introduce appropriate modifications to certain terms of original Fokker-Planck equation at each singular point so that the modified equation has a stationary solution. With the stable stationary solution, the intermediate behavior of the modified Fokker-Planck equation can be captured and described to some extent. The non-modified solutions are compared to modified solutions within finite time and a relatively concrete estimation is given in terms of the modification parameter and the given finite time. We also study some possible modifications. For each modification, the properties of the stationary solution is given. Some numerical results of the time-evolution solutions for these modified equations are also included.

As an attempt, we have initiated in this project the study of the difference between the modified and non-modified stochastic differential equations (SDEs). Although no complete analytical results are available, our initial work appears pointing in a promising direction, based on the numerical simulation results that we have observed. The further study of the SDEs will be carried out in the future work.

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Chapter 1

Introduction

1.1 Brief Description of the Present Project

In this project, a Fokker-Planck equation (FPE) with two singular points is studied. The equation is derived from a stochastic evolution equation, LMM-SABR model, which is widely used in financial industry.

It is difficult to directly study the original equation due to the singularity. As an alternative approach, we introduce appropriate modifications to certain terms of original Fokker-Planck equation at each singular point so that the modified equation has a stationary solution. With the stable stationary solution, the intermediate behavior of the modified Fokker-Planck equation can be captured and described to some extent. The non-modified solutions are compared to modified solutions within finite time and a relatively concrete estimation is given in terms of the modification parameter and the given finite time. We also study some possible modifications. For each modification, the properties of the stationary solution is given. Some numerical results of the time-evolution solutions for these modified equations are also included.

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1.2 Background of the LMM-SABR Model

The concept of interest rates is widely used in our daily life. When one saves money in a banking account, lends money to others or applies for a mortgage loan from a bank, an interest should be paid by the borrower to the lender and the amount is based on the agreed interest rate. Interest rates are normally expressed as a percentage of the principal for a period of one year. Fluctuation of interest rates within a financial market implies a significant risk for investors. To control the uncertainty of the market, more and more advanced mathematical models nowadays are being used in managing interest rate risk.

The LIBOR market model (LMM) is such an interest rate model that is based on the evolution of forward LIBOR rates (London Interbank Offered Rates), which are one of the most important interest rates quoted in the market. The LIBOR market model may be interpreted as a collection of forward LIBOR rates with spanning tenors and maturities. Its dynamics will be introduced in the following sub-section.

Another model in mathematical finance that we want to introduce is called SABR model, where the name stands for "Stochastic, alpha, beta, rho", referring to the parameters of the model. The SABR model is widely used by practitioners in the financial industry, especially in the interest rate derivative markets. It was developed by Patrick Hagan, Deep Kumar, Andrew Lesniewski and Diana Woodward. The SABR model describes a single forward rate such as LIBOR forward rate or a forward swap rate. The details about this model's dynamics will also be introduced in the following sub-section.

LMM-SABR is a model that combines the LMM model and the SABR model in the sense that it extends the LMM model so that it allows for stochastic SABR style volatility of each LIBOR forward rate. In the following section, we will start with the LMM model and the SABR model and then introduce the LMM-SABR model.

1.2.1 LMM-SABR and the Related Models

First we look at the LMM model. Consider a sequence of approximately equally spaced dates $0 = T_0 \leq T_1 \leq T_2 \leq \cdots \leq T_N$. A standard LIBOR forward rate F_j, $j = 0, 1, 2, \cdots, N-1$, is associated with a forward rate agreement (FRA) which starts on T_j and matures on T_{j+1}. (Usually, it is assumed that $N = 120$ and the F_j's are 3 -month LIBOR forward rates.) We model each F_j as a continuous time stochastic process $F_j(t)$, $0 \leq t \leq T$. Under a suitable measure, say T_k-forward measure, the dynamics of the LMM is given by the following system of stochastic differential equations:

$$dF_j(t) = C_j(t) \times \begin{cases} -\Sigma_{i=j+1}^{i=k} \frac{\rho_{ji}\delta_i C_i(t)}{1+\delta_i F_i(t)} dt + dW_j(t), \ j < k; \\ dW_j(t), \ j = k; \\ \Sigma_{i=k+1}^{i=j} \frac{\rho_{ji}\delta_i C_i(t)}{1+\delta_i F_i(t)} dt + dW_j(t), \ j > k. \end{cases} \tag{1.1}$$

where $C_j(t) = C_j(F_j(t), t)$ are instantaneously volatilities and are usually specified as

$$C_j(F_j(t), t) = \sigma_j(t) F_j^{\beta_j}(t), \ 0 \leq \beta_j \leq 1, \tag{1.2}$$

with deterministic functions $\sigma_j(t)$. The instantaneous correlation structure is given by

$$E[dW_j(t) dW_k(t)] = \rho_{jk}(t). \tag{1.3}$$

Under the spot measure, the LMM dynamics reads

$$dF_j(t) = C_j(t) \left(\Sigma_{\gamma(t) \leq i \leq j} \frac{\rho_{ji}\delta_i C_i(t)}{1+\delta_i F_i(t)} dt + dW_j(t) \right), \tag{1.4}$$

equipped with initial values of LIBOR forwards

$$F_j(0) = F_j^0, \tag{1.5}$$

and boundary conditions at $F_j = 0$, where F_j^0 is the current value of the forward which is implied by the current yield curve.

The SABR model attempts to capture the dynamics of a single forward rate F. Depending on the context, this forward rate could be a LIBOR forward, a forward swap rate, the forward yield on a bond, etc. The full dynamics of the SABR model is given by

$$\begin{aligned} dF(t) &= \sigma(t)C(F(t))dW(t) \\ d\sigma(t) &= \alpha\sigma(t)dZ(t), \end{aligned} \tag{1.6}$$

where the two Wiener processes $W(t)$ and $Z(t)$ are in general correlated by

$$E[dW(t)dZ(t)] = rdt \tag{1.7}$$

with a constant coefficient r and α is a positive constant. Usually the diffusion coefficient $C(F)$ is assumed to be of the type $C(F) = F^\beta$ with $0 \leq \beta < 1$ (If $\beta = 1$ and $r > 0$, it fails to be a martingale) and we supplement the dynamics with the initial condition

$$F(0) = F^0,\ \sigma(0) = \sigma^0, \tag{1.8}$$

where F^0 is the current value of the forward and σ^0 is the current value of the β-volatility.

The LMM-SABR model is developed from the LMM model and the SABR model. The process of combining the two models into one and the relevant proofs can be found in [19]. In this project, we are concerned about the dynamics of the equations under the spot measure, which read as

$$\begin{aligned} dF_j(t) &= \sigma_j(t)F_j(t)^{\beta_j}\left(\Sigma_{\gamma(t)\leq i\leq j}\frac{\rho_{ji}\delta_i\sigma_i(t)F_i(t)^{\beta_i}}{1+\delta_iF_i(t)}dt + dW_j(t)\right), \\ d\sigma_j(t) &= \alpha_j(t)\sigma_j(t)\left(\Sigma_{\gamma(t)\leq i\leq j}\frac{r_{ji}\delta_i\sigma_i(t)F_i(t)^{\beta_i}}{1+\delta_iF_i(t)}dt + dZ_j(t)\right), \end{aligned} \tag{1.9}$$

where $\gamma(t) = \frac{t}{T_j}$ and the definitions of other items are the same as above.

1.2.2 A Simple Version of LMM-SABR Model and Its Fokker-Planck Equation (FPE)

Since the SABR model is a term structure model that is quite complex for $N \geq 2$, we start with case $N = 1$ and hope that some of the results of the simple case can be carried onto the more complicated one. In this project, the simple LMM-SABR model will be our focuse. For $N = 1$, the model reads

$$\begin{aligned} dF =& \quad \sigma F^{\beta}\left(\tfrac{\Delta\sigma F^{\beta}}{1+\Delta F}dt + dW\right) \\ d\sigma =& \quad \alpha\sigma\left(\tfrac{r\Delta\sigma F^{\beta}}{1+\Delta F}dt + dZ\right), \end{aligned} \tag{1.10}$$

where $E(dWdZ) = rdt,\ -1 \leq r \leq 1,\ 0 \leq \beta \leq 1$ and α is a positive constant. Note from the second equation of (1.10) that

$$\frac{\Delta\sigma^2 F^{\beta}}{1+\Delta F}dt = \frac{1}{\alpha r}d\sigma - \frac{\sigma}{r}dZ \tag{1.11}$$

Also note that $dZ = rdW + \sqrt{1-r^2}dB$ with dB independent of dW. Then the above equation becomes

$$dF = F^{\beta}\left[\frac{1}{\alpha r}d\sigma - \frac{\sqrt{1-r^2}}{r}\sigma dB\right], \tag{1.12}$$

which implies

$$\int \frac{dF}{F^{\beta}} = \int \frac{d\sigma}{\alpha r} - \int \frac{\sqrt{1-r^2}}{r}\sigma dB \tag{1.13}$$

Since $E(\int \frac{\sqrt{1-r^2}}{r}\sigma dB) = 0$, we assume that the relationship of F and σ is given by $\int \frac{dF}{F^{\beta}} = \int \frac{d\sigma}{\alpha r}$. Then we have

$$\sigma = \frac{\alpha r}{1-\beta}\left(F^{1-\beta} - C_0\right), \tag{1.14}$$

where $C_0 = F_0^{1-\beta} - \frac{1-\beta}{\alpha r}\sigma_0$. Using (1.10) and (1.14), we have

$$dF = \frac{\Delta\left(\frac{\alpha r}{1-\beta}\right)^2 (F - C_0 F^{\beta})^2}{\Delta F + 1}dt + \frac{\alpha r}{1-\beta}(F - C_0 F^{\beta})dW. \tag{1.15}$$

Rewrite equation (1.15) as

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dW(t) \tag{1.16}$$

where

$$C_0 = F_0^{1-\beta} - \frac{1-\beta}{\alpha r}\sigma_0 \tag{1.17}$$

$$\sigma(x) = \frac{\alpha r}{1-\beta}(x - C_0 x^\beta) \tag{1.18}$$

and

$$\mu(x) = \frac{\Delta\sigma^2(x)}{\Delta x + 1} \tag{1.19}$$

with $\Delta > 0$, $\alpha > 0$, $-1 \leq r \leq 1$ and $0 \leq \beta \leq 1$ are all constants.

In application of physics, engineering and finance, the study of such a process that is given by a Stochastic Differential Equation (SDE) is often connected to a Partial Differential Equation (PDE). As known, the two types of PDEs derived from SDEs are Kolmogorov's forward equations or Fokker-Planck equation (FPEs) and backward equations. Given the information about the state x of the system at time t (namely a probability distribution $p_t(x)$), the Fokker-Planck Equation describes the probability distribution of the state at a later time $s > t$. The Kolmogorov backward equation on the other hand is useful when one is interested at time t in whether at a future time $s > t$ the system will be in a given subset of states or the target set.

In this project, we are interested in the time evolution of the probability density function, if exists, of forward interest rates over time period $[0, T]$ and therefore will study the dynamics of the FPE connected to (1.16).

Assuming the probability density of the stochastic process $X(t)$ defined by (1.16) exists under certain conditions of parameters and letting $\rho(x(t)) = \rho(x,t)$ be the density function, then the Fokker-Planck equation reads as

$$\frac{\partial \rho(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2(x)\rho(x,t)\right) - \frac{\partial}{\partial x}\left(\mu(x)\rho(x,t)\right) \tag{1.20}$$

for $t \geq t_0$ with initial condition $\rho(x,t) = \rho_0(x)$.

In review of $\sigma^2(x)$, Fokker-Planck equation (1.16) is degenerate at $x = 0$ and $x = x^* = C_0^{\frac{1}{1-\beta}}$. The degeneracy creates significant difficulties for the study of the equation. The existence of stationary is questionable. Recently, Huang, etc. ([15, 16, 17]) studied the existence and non-existence of steady states of Fokker-Planck Equations in a general domain in R^n with $L^{\bar{p}}_{loc}$ drift term and $W^{1,\bar{p}}_{loc}$ diffusion term for any $\bar{p} > n$. In [17], they also give some existence results of stationary measures of the Fokker-Planck equation under Lyapunov conditions which allow the degeneracy of diffusion. For FPE (1.16), there are two singular distributions centered at $x = 0$ and $x = x^*$, respectively. These singular distributions do not provide much information for intermediate behavior that is the main concerns from practical point of view.

In the present work, we modify the original FPE so that its regular stationary solution exists. Through the stationary solution of the modified equation, the information of the original equation can be captured to some degree. With certain types of modification, difference between the modified solution and the original solution can be estimated within limited time. This would help describe the intermediate behaviors of the non-modified solutions.

Chapter 2

Brief Review of Stochastic Integrals and Stochastic Differential Equations

Ordinal differential equations (ODEs) play a prominent role in many disciplines including engineering, physics, economics, and biology. In a less-than ideal world, differential differential equation models or systems are subject to unknown factors or noises, in which case stochastic integrals and stochastic differential equations are employed to capture effect of the noises.

A stochastic differential equation (SDE) is a differential equation in which one or more of the terms is a stochastic process, resulting in a solution which is itself a stochastic process. Stochastic differential equation are used to model diverse phenomena such as fluctuating stock prices or physical systems subject to thermal fluctuations. In this chapter, we briefly review definitions and some basic properties of stochastic integrals and stochastic differential equations, which are closely related to the present project.

2.1 Stochastic Integrals

In this section, we recall definition of stochastic process and introduce some basic results on stochastic calculus, stochastic integrals and stochastic differential equations (SDEs) (See references [1, 7, 8, 9, 10]).

Definition 2.1.1. Let $\{\Omega, \mathscr{F}, P\}$ be a probability space. A stochastic process with state space S is a collection of random variables $\{X_t \in S, t \in T\}$ defined on $\{\Omega, \mathscr{F}, P\}$, where the set T is called its parameter set.

If $T = \mathbb{R} = \{0, 1, 2, \cdots\}$, the process is said to be a discrete parameter process and if T is not countable, the process is said to have a continuous parameter. In the latter case, the usual examples are $T = \mathbb{R}_+ = [0, \infty)$ and $T = [a, b] \subset \mathbb{R}$. The index t represents time and X_t represents the "state" or the "position" of the process at time t. When the state space $S = \mathbb{R}$, the process is called real-valued. For every $\omega \in \Omega$, the mapping $t \to X_t(\omega)$ defined on the parameter set T, is called a trajectory or sample path of the process.

A real valued stochastic process $\{X_t, t \in T\}$ is said to be Gaussian or normal if its finite-dimensional marginal distributions are multi-dimensional Gaussian laws. The mean $m_X(t)$ and the convariance function $\Gamma_X(x,t)$ of a Gaussian process determin its finite-dimensional marginal distributions. Conversely, suppose that we are given an arbitrary function $m : T \to \mathbb{R}$, and a symmetric function $\Gamma : T \times T \to \mathbb{R}$, which is nonnegative definite, *i.e.*

$$\Sigma_{i,j=1}^{n} \Gamma(t_i, t_j) a_i a_j \geq 0 \tag{2.1}$$

for all $t_i \in T$, $a_i \in \mathbb{R}$ and $n \geq 1$. Then there exists a Gaussian process with mean m and converence function Γ.

Definition 2.1.2. A stochastic process $\{B_t, t \geq 0\}$ is called a Brownian motion (or Winener process) if it satisfies the following conditions:

(i) $B_0 = 0$;

(ii) For all $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, the increments $B_{t_n} - B_{t_{n-1}}, \cdots, B_{t_2} - B_{t_1}$, are independent random variables;

(iii) If $0 \leq s \leq t$, the increment $B_t - B_s$ has the normal distribution $N(0, t-s)$;

(iv) The process $\{B_t\}$ has continuous trajectories.

Brownian motion is a Gaussian process. In fact, the probability distribution of a random vector

$(B_{t_1},\ \cdots, B_{t_n})$, for $0 < t_1 < \cdots < t_n$, is normal because this vector is a linear transformation of the vector $(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})$ which is a joint normal distribution since its components are independent and normal.

Denote by $L^2_{a,T}$ the space of stochastic processes $u = \{u_t, t \in [0,T]\}$ such that:

(a) u is adaped and measurable, i.e. the mapping $(s,\omega) \to u_s(\omega)$ is measurable on the product space $[0,T] \times \Omega$ with respect to the product σ-field $\mathscr{B}_{[0,T]} \times \mathscr{F}$.

(b) $E\left(\int_0^T u_t^2 dt\right) < \infty$.

Definition 2.1.3. A process u in $L^2_{a,T}$ is a simple process if it is of the form

$$u_t = \Sigma_{j=1}^n \phi_j \mathbf{1}_{(t_j, t_{j-1}]}(t), \tag{2.2}$$

where $0 = t_0 < t_1 < \cdots < t_n = T$ and $\phi_j' s$ are square integrable $\mathscr{F}_{t_{j-1}}$-measurable random variables.

Definition 2.1.4. Given a simple process u, the stochastic integral of u with respect to Brownian motion B is defined as

$$\int_0^T u_t dB_t = \Sigma_{j=1}^n \phi_j (B_{t_j} - B_{t_{j-1}}). \tag{2.3}$$

Lemma 2.1.5. *If u is a process in $L^2_{a,T}$, then there exists a squence of simple processes $u^{(n)}$ such that*

$$\lim_{n\to\infty} E\left(\int_0^T |u_t - u_t^{(n)}|^2 dt\right) = 0. \tag{2.4}$$

Definition 2.1.6. The stochastic integral of a process u in $L^2_{a,T}$ is defined as the following limit in mean square

$$\int_0^T u_t dB_t = \lim_{n\to\infty} \int_0^T u_t^{(n)} dB_t \tag{2.5}$$

where $u^{(n)}$ is the approximating sequence of simple processes in the above lemma.

Denote by $L^1_{a,T}$ the space of processes v which satisfies property (a) and

(b') $P\left(\int_0^T |v_t| dt < \infty\right) = 1$.

Definition 2.1.7. A continuous and adapted stochastic process $\{X_t, 0 \le t \le T\}$ is called an Ito process if it can be expressed in the form

$$X_t = X_0 + \int_0^t u_s dB_s + \int_0^t v_s ds, \tag{2.6}$$

where u belongs to the space $L^2_{a,T}$ and v belongs to the space $L^1_{a,T}$.

In differential notation, the above equation writes

$$dX_t = u_t dB_t + v_t dt. \tag{2.7}$$

Theorem 2.1.8. (Ito formula) *Suppose that X is an Ito process of the form above. Let $f(t,x)$ be a function of twice differentiable with respect to the variable x and once differentiable with respect to t, with continuous partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial f}{\partial t}$. Then the process $Y_t = f(t,X_t)$ is again an Ito process with the representation*

$$\begin{aligned} Y_t = \; & f(0,X_0) + \int_0^t \frac{\partial f}{\partial t}(s,X_s)ds + \int_0^t \frac{\partial f}{\partial x}(s,X_s)u_s dB_s + \int_0^t \frac{\partial f}{\partial x}(s,X_s)v_s dB_s \\ & + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,X_s)u_s^2 ds. \end{aligned} \tag{2.8}$$

Remark 2.1.1. In differential notation, Ito formula can be written as

$$df(t,X_t) = \frac{\partial f}{\partial t}(t,X_t)dt + \frac{\partial f}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x,X_s)(dX_t)^2, \tag{2.9}$$

where $(dX_t)^2$ is computed using the product rule

$$(dt)^2 = dtdB_t = dB_t dt = 0 \text{ and } (dB_t)^2 = dt. \tag{2.10}$$

The process Y_t is an Ito process with the representation

$$Y_t = Y_0 + \int_0^t \tilde{u}_s dB_s + \int_0^t \tilde{v}_s ds, \tag{2.11}$$

where

$$
\begin{aligned}
Y_0 = &\quad f(0, X_0), \\
\tilde{u}_t = &\quad \frac{\partial f}{\partial x}(t, X_t) u_t, \\
\tilde{v}_t = &\quad \frac{\partial f}{\partial t}(t, X_t) + \frac{\partial f}{\partial x}(t, X_t) v_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, X_t) u_t^2.
\end{aligned} \tag{2.12}
$$

Notice that $\tilde{u}_t \in L^2_{a,T}$ and $\tilde{v}_t \in L^1_{a,T}$ due to the continuity of X_t.

2.2 Stochastic Differential Equations (SDEs)

Let $\{B_t, t \geq 0\}$ be a Brownian motion process. An equation of the form

$$dX_t = a(X_t, t)dt + b(X_t, t)dB_t, \tag{2.13}$$

where the given functions $a(x,t)$ and $b(x,t)$ are called respectively drift and diffusion and $\{X_t, t \geq 0\}$ is the unknown process, is called a stochastic differential equation (SDE) driven by Brownian motion $\{B_t, t \geq 0\}$.

Definition 2.2.1. A process $\{X_t, t \geq 0\}$ is called a solution of the above SDE if for all $t > 0$, the integral $\int_0^t a(X_s, s)ds$ and $\int_0^t b(X_s, s)dB_s$ exist and X_t is an Ito process with

$$X_t = X_0 + \int_0^t a(X_t, s)ds + \int_0^t b(X_s, s)dB_s. \tag{2.14}$$

Theorem 2.2.2. (Existence and Uniqueness) *For a fixed $T > 0$, suppose that $a(x,t)$, $b(x,t)$: $\mathbb{R} \times [0,T] \to \mathbb{R}$ are continuous and there exists a constant $L > 0$ such that*

$$|a(x,t) - a(\hat{x},t)| \leq L|x - \hat{x}| \tag{2.15}$$

$$|b(x,t) - b(\hat{x},t)| \leq L|x - \hat{x}| \tag{2.16}$$

$$|a(x,t)| \leq L(1 + |x|) \tag{2.17}$$

$$|b(x,t)| \leq L(1+|x|) \tag{2.18}$$

for all $0 \leq t \leq T$, $x, \hat{x} \in \mathbb{R}$. *Let* X_0 *be any* $\mathbb{R}$*-valued random variable such that* $E(|X_0|^2) < \infty$ *and* X_0 *is independent of Brownian motion* $\{B_t, 0 \leq t \leq T\}$. *Then there exists a unique solution* $X \in L^2_{0,T}$ *of stochastic differential equation*

$$\begin{cases} dX_t & = a(X_t,t)dt + b(X_t,t)dB_t, \ (0 \leq t \leq T), \\ X(0) & = X_0. \end{cases} \tag{2.19}$$

Remark 2.2.1. Uniqueness means that if $X_t, \hat{X}_t \in L^2_{0,T}$, with continuous sample paths almost surely and both solve the above SDE, then

$$P(X_t = \hat{X}_t \text{ for all } 0 \leq t \leq T) = 1. \tag{2.20}$$

Chapter 3

Fokker-Planck Equations

The Fokker-Planck equation is a partial differential equation that describes the time evolution of the probability density function for a Brownian motion.

Let $X_t, t \geq 0$ be the process defined by (2.13) and $p(x,t)$ be the transition probability density function for X_{t+s} given $X_s = y$. Specifically p is defined by

$$\begin{aligned} \int_A p(x,t|y,s)dx &= \Pr[X_{t+s} \in A | X_s = y] \\ &= \Pr[X_t \in A | X_s = 0]. \end{aligned} \tag{3.1}$$

It turns out that $p(x,t) = p(x,t|y,s)$ satisfies a deterministic PDE, Fokker-Planck equation,

$$\frac{\partial p(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2[b^2(x)p(x,t)]}{\partial x^2} - \frac{\partial[a(x)p(x,t)]}{\partial x}. \tag{3.2}$$

In this chapter we briefly review derivation of the Fokker-Planck equation from a given SDE and some of the basic properties of the FPEs as well.

3.1 Derivation of Fokker-Planck Equation

Consider a differentiable function $V(x,t)$ with $V(X_t,t) = 0$ for $t \notin (0,T)$. Then by Ito's Lemma,

$$\begin{aligned} dV(X_t,t) = & \left[\tfrac{\partial V(X_t,t)}{\partial t} + a(X_t)\tfrac{\partial V(X_t,t)}{\partial x} + \tfrac{1}{2}b^2(X_t)\tfrac{\partial^2 V(X_t,t)}{\partial^2 x}\right] dt \\ & + \left[b(X_t)\tfrac{\partial V(X_t,t)}{\partial x}\right] dW_t. \end{aligned} \tag{3.3}$$

For any $T > 0$, this is equivalent to

$$\begin{aligned} & V(X_T,T) - V(X_0,0) \\ = & \int_0^T \left[\tfrac{\partial V(X_t,t)}{\partial t} + a(X_t)\tfrac{\partial V(X_t,t)}{\partial x} + \tfrac{1}{2}b^2(X_t)\tfrac{\partial^2 V(X_t,t)}{\partial^2 x}\right] dt + \int_0^T \left[b(X_t)\tfrac{\partial V(X_t,t)}{\partial x}\right] dW_t. \end{aligned} \tag{3.4}$$

Taking the conditional expection on both sides of the above equation, given X_0, we have

$$\begin{aligned} & E[V(X_T,T) - V(X_0,0)] \\ = & E\int_0^T \left[\tfrac{\partial V(X_t,t)}{\partial t} + a(X_t)\tfrac{\partial V(X_t,t)}{\partial x} + \tfrac{1}{2}b^2(X_t)\tfrac{\partial^2 V(X_t,t)}{\partial^2 x}\right] dt + E\int_0^T \left[b(X_t)\tfrac{\partial V(X_t,t)}{\partial x}\right] dW_t \\ = & \int_{\mathbb{R}}\int_0^T \tfrac{\partial V(x,t)}{\partial t} dt\, p(x,t|y,s)dx + \int_{\mathbb{R}}\int_0^T a(x)\tfrac{\partial V(x,t)}{\partial x} dt\, p(x,t|y,s)dx \\ & + \tfrac{1}{2}\int_{\mathbb{R}}\int_0^T b^2(x)\tfrac{\partial^2 V(x,t)}{\partial^2 x} dt\, p(x,t|y,s)dx \\ = & I_1 + I_2 + I_3 \end{aligned} \tag{3.5}$$

where $E\int_0^T \left[b(X_t)\frac{\partial V(X_t,t)}{\partial x}\right] dW_t = 0$ since $E[dW_t] = 0$. The three integrals can be evaluated as follows. Note that

$$\int_0^T \frac{\partial V(x,t)}{\partial t} p(x,t|y,s)dt = V(x,t)p(x,t|y,s)\big|_0^T - \int_0^T \frac{\partial p(x,t|y,s)}{\partial t} V(x,t)dt \tag{3.6}$$

and $V(x,T) = V(x,0) = 0$. Then we have

$$I_1 = -\int_{\mathbb{R}}\int_0^T \frac{\partial p(x,t|y,s)}{\partial t} V(x,t)dt. \tag{3.7}$$

For the second integral, we have

$$\int_{\mathbb{R}}\int_0^T a(x)\frac{\partial V(x,t)}{\partial x}dt\,p(x,t|y,s)dx = \int_0^T\int_{\mathbb{R}} a(x)\frac{\partial V(x,t)}{\partial x}p(x,t|y,s)dxdt \tag{3.8}$$

and

$$\int_{\mathbb{R}} a(x)\frac{\partial V(x,t)}{\partial x}p(x,t|y,s)dx = a(x)V(x,t)p(x,t|y,s)\big|_{-\infty}^{\infty} - \int_{\mathbb{R}} V(x,t)\frac{\partial[a(x)p(x,t|y,s)]}{\partial x}dx. \tag{3.9}$$

With $p(\infty,t|y,s) = p(-\infty,t|y,s) = 0$ we have

$$\begin{aligned} I_2 &= -\int_0^T\int_{\mathbb{R}} V(x,t)\tfrac{\partial[a(x)p(x,t|y,s)]}{\partial x}dx \\ &= -\int_{\mathbb{R}}\int_0^T V(x,t)\tfrac{\partial[a(x)p(x,t|y,s)]}{\partial x}dx. \end{aligned} \tag{3.10}$$

Now we consider the evaluation of integral I_3. Note that

$$\begin{aligned} &\int_{\mathbb{R}} b^2(x)p(x,t|y,s)\tfrac{\partial^2 V(x,t)}{\partial x^2}dx \\ = \;& b^2(x)p(x,t|y,s)\tfrac{\partial V(x,t)}{\partial x}\big|_{-\infty}^{\infty} - \int_{\mathbb{R}}\tfrac{\partial[b^2(x)p(x,t|y,s)]}{\partial x}\tfrac{\partial V(x,t)}{\partial x}dx \\ = \;& b^2(x)p(x,t|y,s)\tfrac{\partial V(x,t)}{\partial x}\big|_{-\infty}^{\infty} - \tfrac{\partial[b^2(x)p(x,t|y,s)]}{\partial x}V(x,t)\big|_{-\infty}^{\infty} \\ & + \int_{\mathbb{R}} V(x,t)\tfrac{\partial^2[b^2(x)p(x,t|y,s)]}{\partial x^2}dx. \end{aligned} \tag{3.11}$$

Again with $p(\infty,t|y,s) = p(-\infty,t|y,s) = 0$ and $p_x(\infty,t|y,s) = p_x(-\infty,t|y,s) = 0$ we have

$$\int_{\mathbb{R}} b^2(x)p(x,t|y,s)\frac{\partial^2 V(x,t)}{\partial x^2}dx = \int_{\mathbb{R}} V(x,t)\frac{\partial^2[b^2(x)p(x,t|y,s)]}{\partial x^2} \tag{3.12}$$

and thus

$$\begin{aligned} I_3 &= \tfrac{1}{2}\int_{\mathbb{R}}\int_0^T b^2(x)\tfrac{\partial^2 V(x,t)}{\partial^2 x}dt\,p(x,t|y,s)dx \\ &= \tfrac{1}{2}\int_0^T\int_{\mathbb{R}} b^2(x)\tfrac{\partial^2 V(x,t)}{\partial^2 x}dt\,p(x,t|y,s)dx \\ &= \tfrac{1}{2}\int_0^T\int_{\mathbb{R}} V(x,t)\tfrac{\partial^2[b^2(x)p(x,t|y,s)]}{\partial x^2} \\ &= \tfrac{1}{2}\int_{\mathbb{R}}\int_0^T V(x,t)\tfrac{\partial^2[b^2(x)p(x,t|y,s)]}{\partial x^2}. \end{aligned} \tag{3.13}$$

Therefore, with $p(x,t) = p(x,t|y,s)$, we have

$$\begin{aligned} & E[V(X_T,T) - V(X_0,0)] \\ = \quad & \int_{\mathbb{R}} \int_0^T V(x,t) \left[-\frac{\partial p(x,t)}{\partial t} - \frac{\partial [a(x)p(x,t)]}{\partial x} + \frac{1}{2} \frac{\partial^2 [b^2(x)p(x,t)]}{\partial x^2} \right] dt dx. \end{aligned} \tag{3.14}$$

On the other hand, $V(X_T,T) = V(X_0,0) = 0$ implies $E[V(X_T,T) - V(X_0,0)] = 0$. This leads to

$$-\frac{\partial p(x,t)}{\partial t} - \frac{\partial [a(x)p(x,t)]}{\partial x} + \frac{1}{2} \frac{\partial^2 [b^2(x)p(x,t)]}{\partial x^2} = 0 \tag{3.15}$$

or the Fokker-Planck equation

$$\frac{\partial p(x,t)}{\partial t} = \frac{1}{2} \frac{\partial^2 [b^2(x)p(x,t)]}{\partial x^2} - \frac{\partial [a(x)p(x,t)]}{\partial x}. \tag{3.16}$$

3.2 Basic properties of Fokker-Planck Equation

3.2.1 Initial Conditions and Boundary Conditions

For a general Fokker-Planck equation in form of

$$p_t(x,t) = \frac{1}{2}[b^2(x)p(x,t)]_{xx} - [a(x)p(x,t)]_x \tag{3.17}$$

with $a(x) \neq 0$ and $b(x) \neq 0$ in a rectangular area, the maximum principle asserts that the maximum of $p(x,t)$ can't be assumed anywhere inside the rectangle but only on the bottom where the time initials or the lateral sides (unless p is a constant). The minimum value has the same property. It can be attained only on the bottom or the lateral sides. To prove the minimum principle, we just need to apply the maximum principle to $-p(x,t)$.

Theorem 3.2.1. (Maximum Principle) *Assuming $a(x) \neq 0$, $b(x) \neq 0$ and they both are differentiable on a rectangle $R = \{0 \leq x \leq l, 0 \leq t \leq T\}$ in space-time, If $p(x,t)$ satisfies the above Fokker-Planck equation, then $p(x,t=0) \geq 0$ implies $p(x,t) \geq 0$ for $0 < t \leq T$.*

Proof. Let

$$t_0 = \min\{t > 0 : p(x,t) = 0, \text{for some } x_0 \in [0,l]\}, \tag{3.18}$$

then we have $p(x_0,t_0) = 0$. Since $p(x,t=0) \geq 0$ and $p(x,t=t_0)$ is continuously differentiable with respect to x, we conclude that (x_0,t_0) is a minimum of $p(x,t_0)$ and furthermore

$$p_x(x_0,t_0) = 0, \tag{3.19}$$

$$p_{xx}(x_0,t_0) \geq 0. \tag{3.20}$$

Consider

$$\begin{aligned} p_t(x,t) &= \tfrac{1}{2}[b^2(x)p(x,t)]_{xx} - [a(x)p(x,t)]_x \\ &= \tfrac{1}{2}[2(b'(x))^2 p(x,t) + 2b(x)b''(x)p(x,t) + 4b(x)b'(x)p_x(x,t) + b^2(x)p_{xx}] \\ &\quad -a(x)p_x(x,t) - a'(x)p(x,t). \end{aligned} \tag{3.21}$$

Given $p(x_0,t_0) = 0$ and $p_x(x_0,t_0) = 0$, we have

$$p_t(x_0,t_0) = \frac{1}{2}b^2(x_0)p_{xx}(x_0,t_0). \tag{3.22}$$

With the assumption $b(x) \neq 0$ on R, the above equation gives

$$p_t(x_0,t_0) \geq 0, \tag{3.23}$$

which implies that $p(x_0,t_0) \geq 0$ for $t > 0$. This process can be repeated at other points where $p(x,t)$ hits zero and then it concludes that the solution $p(x,t)$ to the Fokker-Planck equation never goes negative on R, given the initial condition $p(x,t=0) \geq 0$. $\square$

3.2.2 Stationary Solution

Given one-dimensional Fokker-Planck equation in the following form

$$\frac{\partial p(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2[b^2(x)p(x,t)]}{\partial x^2} - \frac{\partial[a(x)p(x,t)]}{\partial x}, \tag{3.24}$$

if the stationary solution $p(x,\infty)$ exists, it should satisfy

$$\frac{1}{2}\frac{d^2[b^2(x)p(x,\infty)]}{dx^2} = \frac{d[a(x)p(x,\infty)]}{dx}, \tag{3.25}$$

which leads to

$$\frac{1}{2}\frac{d}{dx}\left[b^2(x)p(x,\infty)\right] = a(x)p(x,\infty) + \frac{1}{2}c_1, \tag{3.26}$$

where c_1 is an integral constant. Multiplying on both sides of the above equation by integrating factor

$$g(x) = \exp\left(-\int_{x_0}^{x}\left[\frac{2a(y)}{b^2(y)}\right]dy\right), \tag{3.27}$$

we have

$$\frac{d}{dx}\left[b^2(x)p(x,\infty)g(x)\right] = c_1 g(x). \tag{3.28}$$

Therefore, the stationary solution has the general form

$$p(x,\infty) = c_1\frac{\int_{x_0}^{x} g(y)dy}{g(x)b^2(x)} + c_2\frac{1}{g(x)b^2(x)}, \tag{3.29}$$

where constants c_1 and c_2 are chosen so that $p(x,\infty)$ is non-negative and

$$\int_{-\infty}^{\infty} p(x,\infty)dx = 1 \tag{3.30}$$

or

$$\int_{0}^{\infty} p(x,\infty)dx = 1. \tag{3.31}$$

Chapter 4

The Degenerate Fokker-Planck Equation

As mentioned early, the focus of the present project is the modified FPE and its solutions. Since the stationary solution of the modified equation exists, the convergence rate of the time-dependent solutions depends on the modification parameter ε. On the other side, given finite time T, we are interested in the intermediate behavior of the time-dependent solutions for $t < T$.

In general, it's expected that for a fixed modification parameter ε, large T gives better convergence rate. For the comparison between the time-dependent solution and the stationary solution, small ε is better. Therefore, given T fixed, ε can not be too small.

In this chapter, we discuss under what conditions of parameters, the modified solutions can give a relatively good estimate estimation of the solutions for the original equation. We also attempts to study the intermediate behaviors of the time-dependent solutions for $t < T$.

4.1 Introduction to the Degenerate Fokker-Planck Equation

As briefly introduced in the first chapter, the Fokker-Planck equation describes the evolution of the probability density, assuming it exists under certain condition of parameters. The equation can be written as follows

$$\frac{\partial \rho(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2(x)\rho(x,t)\right) - \frac{\partial}{\partial x}\left(\mu(x)\rho(x,t)\right) \tag{4.1}$$

for $t \geq t_0$ with initial condition $\rho(x,0) = \rho_0(x)$, where $C_0 = F_0^{1-\beta} - \frac{1-\beta}{\alpha r}\sigma_0$ and

$$\sigma(x) = \frac{\alpha r}{1-\beta}(x - C_0 x^{\beta}), \tag{4.2}$$

and

$$\mu(x,\varepsilon) = \frac{\Delta\sigma^2(x,\varepsilon)}{\Delta x + 1} = \frac{\sigma^2(x,\varepsilon)}{x+\delta} \tag{4.3}$$

with $\Delta > 0$, $\delta = \frac{1}{\Delta}$, $\alpha > 0$, $-1 \leq r \leq 1$ and $0 \leq \beta \leq 1$ are all constants.

Note that (4.1) degenerates at two singular points $x_* = 0$ and $x^* = (C_0)^{\frac{1}{1-\beta}}$ for $C_0 > 0$ since $\sigma(x_*) = \sigma(x^*) = 0$. We intend to modify $\sigma(x)$ to some degree so that the stationary solution is defined and then hope it still captures certain dynamics of the non-modified equation (4.1). Now we consider the following equation with modification

$$\frac{\partial \rho(x,t,\varepsilon)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2(x,\varepsilon)\rho(x,t,\varepsilon)\right) - \frac{\partial}{\partial x}\left(\mu(x,\varepsilon)\rho(x,t,\varepsilon)\right) \tag{4.4}$$

where $\sigma^2(x,\varepsilon) \neq 0$ for any $x \in \mathbb{R}$, $\mu(x,\varepsilon) = \frac{\sigma^2(x,\varepsilon)}{x+\delta}$ with $0 < \varepsilon << 1$ and the other parameters are the same as in (4.1).

With the restrictions above, the stationary solution of (4.4) is well defined as as follows

$$\rho(x,\infty,\varepsilon) = c_1 \frac{\int_{x_0}^{x} g(y,\varepsilon)dy}{g(x,\varepsilon)\sigma^2(x,\varepsilon)} + c_2 \frac{1}{g(x,\varepsilon)\sigma^2(x,\varepsilon)}, \tag{4.5}$$

where constants c_1 and c_2 are chosen so that $\rho(x,\infty,\varepsilon)$ is non-negative. We also require

$$\int_0^{\infty} \rho(x,\infty,\varepsilon)dx = 1 \tag{4.6}$$

or

$$\int_0^{L} \rho(x,\infty,\varepsilon)dx = 1 \tag{4.7}$$

for $0 < L < \infty$.

Remark 4.1.1. If the above integral is up to ∞, we need to first let $C_2 = 0$ and "cut" $\frac{\mu(x,\varepsilon)}{\sigma^2(x,\varepsilon)} = \frac{1}{1+\delta}$ at the far end so that it decays fast enough (see Chapter 5 for details). Our main focus is finite interval and finite boundary from the practical perspective. Therefore we assume $x \in [0,L]$ and $x \in [0,T]$ for some finite $L > 0$ and $T > 0$.

Assuming that all the required conditions are satisfied and the stationary solution has been found, we would like to see next what can be expected on the intermediate behavior of the time-dependent solution. In the following sections, we discuss the required boundary conditions and how well the modified equation can be used to estimate the non-modified equation, given appropriate boundary conditions. From there, useful information can be obtained as indication for selecting modification parameters.

4.2 Robin Boundary Conditions

We are interested in the characteristics of the solutions to the given equation within finite time $0 < T < \infty$. Without loss of generality, let $x = 0$ be the lower bound and $x = L$ be the upper bound. We look for desired boundary conditions such that all the time-dependent solutions are probability density like functions. For the modified FPE, we have the following proposition.

Proposition 4.2.1. *For any fixed L, if $\rho(x,t,\varepsilon)$ is a solution of modified FPE* (4.4) *with Robin boundary conditions*

$$\frac{1}{2}[\sigma^2(x,\varepsilon)\rho(x,t,\varepsilon)]_x(x=0) - [\mu(x,\varepsilon)\rho(x,t,\varepsilon)](x=0) = 0$$

$$\frac{1}{2}[\sigma^2(x,\varepsilon)\rho(x,t,\varepsilon)]_x(x=L) - [\mu(x,\varepsilon)\rho(x,t,\varepsilon)](x=L) = 0,$$

then $\rho(x,t,\varepsilon)$ satisfies

$$\int_0^L \rho(x,t,\varepsilon)dx = \int_0^L \rho(x,0,\varepsilon)dx.$$

Proof. For any fixed $L > 0$, we have

$$\frac{\partial}{\partial t}\left[\int_0^L \rho(x,t,\varepsilon)dx\right] = \int_0^L \rho_t(x,t,\varepsilon)dx.$$

Given modified FPE (4.4), the right hand side of the above equation becomes

$$\int_0^L \rho_t(x,t,\varepsilon)dx = \frac{1}{2}[\sigma^2(x,\varepsilon)\rho(x,t,\varepsilon)]_x - [\mu(x,\varepsilon)\rho(x,t,\varepsilon)]|_0^L.$$

With the Robin boundary conditions, $\int_0^L \rho_t(x,t,\varepsilon)dx = 0$ and therefore

$$\frac{\partial}{\partial t}\left[\int_0^L \rho(x,t,\varepsilon)dx\right] = 0,$$

which means that $\int_0^L \rho(x,t,\varepsilon)dx$ doesn't change as t varies in $[0,T]$. Then we have

$$\int_0^L \rho(x,t,\varepsilon)dx = \int_0^L \rho(x,0,\varepsilon)dx$$

and the proof is complete. $\square$

Proposition 4.2.2. *For any fixed $L > 0$, the modified FPE (4.4) has a probability density like stationary solution $\rho(x,\varepsilon)$ such that $\int_0^L \rho(x,\varepsilon)dx = 1$.*

Proof. In the general form of the stationary solution (4.5), using $\mu(x,\varepsilon) = \frac{\sigma^2(x,\varepsilon)}{x+\delta}$, we have

$$g(x,\varepsilon) = \frac{\delta^2}{(x+\delta)^2}$$

and

$$\int_0^x g(y,\varepsilon)dy = \delta - \frac{\delta^2}{x+\delta}.$$

Thus the stationary solution becomes

$$\rho(x,\varepsilon) = \frac{C_1 x(x+\delta)}{\delta\sigma^2(x,\varepsilon)} + \frac{C_2(x+\delta)}{\delta^2\sigma^2(x,\varepsilon)} = \frac{(x+\delta)(\alpha_1 x+\alpha_2)}{\sigma^2(x,\varepsilon)} \tag{4.8}$$

for some α_1 and α_2. This gives

$$\begin{aligned} \tfrac{1}{2}[\sigma^2(x,\varepsilon)\rho(x,\varepsilon)]_x - \tfrac{\sigma^2(x,\varepsilon)}{x+\delta}\rho(x,\varepsilon) &= \tfrac{1}{2}[(x+\delta)(\alpha_1 x+\alpha_2)]_x - (\alpha_1 x+\alpha_2) \\ &= \tfrac{1}{2}(\delta\alpha_1 - \alpha_2). \end{aligned} \tag{4.9}$$

If $\alpha_2 = \delta\alpha_1$, then

$$\frac{1}{2}[\sigma^2(x,\varepsilon)\rho(x,\varepsilon)]_x - \mu(x,\varepsilon)\rho(x,\varepsilon) = 0 \tag{4.10}$$

at any x.

Now we choose $\alpha_1 = \alpha$ and $\alpha_2 = \delta\alpha$, where α is determined by $\int_0^L \rho(x,\varepsilon)dx = 1$. From

$$\rho(x,\varepsilon) = \frac{(x+\delta)(\alpha x+\delta\alpha)}{\sigma^2(x,\varepsilon)} = \alpha\frac{(x+\delta)^2}{\sigma^2(x,\varepsilon)},$$

one gets

$$\alpha = \alpha(L,\varepsilon) = \left(\int_0^L \frac{(x+\delta)^2}{\sigma^2(x,\varepsilon)}dx\right)^{-1}$$

and $\int_0^L \rho(x,\varepsilon)dx = 1$, which completes the proof. □

Remark 4.2.1. The above stationary solution is independent of L but its domain depends on L. If we choose $L = \infty$, then we will have $\int_0^L \frac{(x+\delta)^2}{\sigma^2(x,\varepsilon)}dx = \infty$ or $\alpha = 0$.

In the following sections, we will study the time-dependent solution of the Robin boundary value problem

$$\begin{cases} \rho_t(x,t,\varepsilon) = [\sigma^2(x,\varepsilon)\rho_t(x,t,\varepsilon)]_{xx} - [\mu(x,\varepsilon)\rho_t(x,t,\varepsilon)]_x \\ \frac{1}{2}\left[\sigma^2(x,\varepsilon)\rho(x,t,\varepsilon)\right]_x - \mu(x,\varepsilon)\rho(x,t,\varepsilon)|_0^L = 0, \end{cases} \tag{4.11}$$

with initial function $\rho(x,0,\varepsilon) = \rho(x,0) \in \mathbb{P}_L$ for all $\varepsilon \geq 0, x \in [0,L]$ and $t \in [0,T]$ where

$$\mathbb{P}_L = \left\{p : \int_0^L p(x,t)dx = 1 \text{ and } p(x,t) \geq 0\right\}.$$

4.3 A Special Transformation

In this section, we use a special transformation and rewrite the boundary value problem (4.11) equivalently in a simple form. From there, the modified and non-modified FPEs are studied and difference of their solutions are estimated in the following sections. The results of the estimation using the transformation can be easily translated back to solution of (4.11).

Noting that $\mu(x,\varepsilon)=\frac{\sigma^2(x,\varepsilon)}{x+\delta}$, we can rewrite equation (4.4) using the transformation

$$u(x,t,\varepsilon)=\frac{\sigma^2(x,\varepsilon)}{x+\delta}\rho(x,t,\varepsilon), \tag{4.12}$$

and have

$$\frac{x+\delta}{\sigma^2(x,\varepsilon)}u_t(x,t,\varepsilon)=\frac{1}{2}\left[(x+\delta)\,u(x,t,\varepsilon)\right]_{xx}-u_x(x,t,\varepsilon). \tag{4.13}$$

Proposition 4.3.1. *$u(x,t,\varepsilon)$ satisfies*

$$u_t(x,t,\varepsilon)=\frac{1}{2}\sigma^2(x,\varepsilon)u_{xx}(x,t,\varepsilon) \tag{4.14}$$

and the new Robin boundary conditions

$$u_x(0,t,\varepsilon)-\frac{1}{\delta}u(0,t,\varepsilon)=0,\ \ u_x(L,t,\varepsilon)-\frac{1}{L+\delta}u(L,t,\varepsilon)=0. \tag{4.15}$$

Proof. The right hand side of equation (4.13) can be simplified as

$$\begin{aligned}
& \tfrac{1}{2}\left[(x+\delta)\,u(x,t,\varepsilon)\right]_{xx}-u_x(x,t,\varepsilon)\\
= & \tfrac{1}{2}\left[u(x,t,\varepsilon)+(x+\delta)\,u_x(x,t,\varepsilon)\right]_{x}-u_x(x,t,\varepsilon)\\
= & \tfrac{1}{2}\left[2u_x(x,t,\varepsilon)+(x+\delta)\,u_{xx}(x,t,\varepsilon)\right]-u_x(x,t,\varepsilon)\\
= & \tfrac{1}{2}(x+\delta)\,u_{xx}(x,t,\varepsilon).
\end{aligned} \tag{4.16}$$

Thus equation (4.4) becomes

$$\frac{x+\delta}{\sigma^2(x,\varepsilon)}u_t(x,t,\varepsilon)=\frac{1}{2}(x+\delta)\,u_{xx}(x,t,\varepsilon) \tag{4.17}$$

or

$$u_t(x,t,\varepsilon)=\frac{1}{2}\sigma^2(x,\varepsilon)u_{xx}(x,t,\varepsilon). \tag{4.18}$$

The Robin boundary conditions in (4.11) under the transformation becomes

$$\frac{1}{2}[(x+\delta)u(x,t,\varepsilon)]_x-u(x,t,\varepsilon)=\frac{1}{2}(x+\delta)u_x(x,t,\varepsilon)-\frac{1}{2}u(x,t,\varepsilon).$$

We have

$$\frac{1}{2}\delta u_x(0,t,\varepsilon)-\frac{1}{2}u(0,t,\varepsilon)=0 \text{ and } \frac{1}{2}(L+\delta)u_x(L,t,\varepsilon)-\frac{1}{2}u(L,t,\varepsilon)=0$$

or equivalently

$$u_x(0,t,\varepsilon)-\frac{1}{\delta}u(0,t,\varepsilon)=0 \text{ and } u_x(L,t,\varepsilon)-\frac{1}{L+\delta}u(L,t,\varepsilon)=0.$$

This completes the proof. $\square$

Therefore the original (non-modified) and the modified FPEs with Robin boundary conditions in terms of u, are given by

$$\begin{cases} u_t(x,t)=\frac{1}{2}\sigma^2(x)u(x,t) \\ u_x(0,t)-\frac{1}{\delta}u(0,t)=0, \;\; u_x(L,t)-\frac{1}{L+\delta}u(L,t)=0 \\ u(x,0)=\phi(x); \end{cases} \tag{4.19}$$

and

$$\begin{cases} u_t(x,t,\varepsilon) = \frac{1}{2}\sigma^2(x,\varepsilon)u(x,t,\varepsilon) \\ u_x(0,t,\varepsilon) - \frac{1}{\delta}u(0,t,\varepsilon) = 0, \;\; u_x(L,t,\varepsilon) - \frac{1}{L+\delta}u(L,t,\varepsilon) = 0 \\ u(x,0,\varepsilon) = \phi(x) \end{cases} \tag{4.20}$$

with some initial function $\phi(x)$. In the following section we will work on (4.19) and (4.20) and estimate the difference of their solutions over finite time and finite domain.

4.4 Difference of the Solutions between (4.19) and (4.20)

Now we compare the two systems and estimate the difference of their solutions. By subtracting the first equation in (4.19) from the one in (4.20), we have

$$\begin{aligned} [u(x,t,\varepsilon) - u(x,t)]_t = \;& \tfrac{1}{2}\sigma^2(x,\varepsilon)u_{xx}(x,t,\varepsilon) - \tfrac{1}{2}\sigma^2(x,\varepsilon)u_{xx}(x,t) \\ & + \tfrac{1}{2}\sigma^2(x,\varepsilon)u_{xx}(x,t) - \tfrac{1}{2}\sigma^2(x)u_{xx}(x,t). \end{aligned} \tag{4.21}$$

Let $v(x,t,\varepsilon) = u(x,t,\varepsilon) - u(x,t)$. Then the above equation becomes

$$v_t(x,t,\varepsilon) = \frac{1}{2}\sigma^2(x,\varepsilon)v_{xx}(x,t,\varepsilon) + \frac{1}{2}[\sigma^2(x,\varepsilon) - \sigma^2(x)]u_{xx}(x,t). \tag{4.22}$$

From the boundary conditions of (4.19) and (4.20), it's easy to verify that v satisfies the following boundary conditions

$$v_x(0,t,\varepsilon) - \frac{1}{\delta}v(0,t,\varepsilon) = 0 \;\text{ and }\; v_x(L,t,\varepsilon) - \frac{1}{\delta + L}v(L,t,\varepsilon) = 0. \tag{4.23}$$

We need the following lemma for our estimation of the difference between the two systems.

Lemma 4.4.1. (Theorem 2.9 in [6]) *Let $\tilde{u}(x,t)$ be the solution of* (4.19) *with $\tilde{u}(x,0) \in H^2$, then $\max_{Q_T} |\tilde{u}_{xx}(x,t)| \leq C(T)|\tilde{u}(x,0)|_{H^2}$ for some $C(T) > 0$, where $Q_T = [0,L] \times [0,T]$.*

Theorem 4.4.2. *If $\pi\delta > x^* = C_0^{\frac{1}{1-\beta}}$ and $L \in (x^*, \pi\delta)$, then for fixed $T > 0$ and $0 << \varepsilon < 1$, there* *exist* $C(T) > 0$ *and* $\eta(\varepsilon) > 0$ *such that*

$$\int_0^L \frac{v^2(x,t,\varepsilon)}{\sigma^2(x,\varepsilon)} dx \le C(T) \|u(x,0)\|_{H^2} \eta(\varepsilon) t \tag{4.24}$$

and $\eta(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

Proof. Let $\bar{\lambda}(x,\varepsilon) = 1 - \frac{\sigma^2(x)}{\sigma^2(x,\varepsilon)}$ with $\sigma^2(x,\varepsilon) > 0$ and apply $\int_0^L \frac{v(x,t,\varepsilon)}{\sigma^2(x,\varepsilon)} \cdot dx$ on both sides of equation (4.22). Then we have

$$\begin{aligned} & \int_0^L \tfrac{v(x,t,\varepsilon)}{\sigma^2(x,\varepsilon)} v_t(x,t,\varepsilon) dx \\ = \; & \tfrac{1}{2} \int_0^L v_{xx}(x,t,\varepsilon) v(x,t,\varepsilon) dx + \tfrac{1}{2} \int_0^L \bar{\lambda}(x,\varepsilon) u_{xx}(x,t) v(x,t,\varepsilon) dx. \end{aligned} \tag{4.25}$$

Note that, on the left hand side of the above equation,

$$\int_0^L \frac{v(x,t,\varepsilon)}{\sigma^2(x,\varepsilon)} v_t(x,t,\varepsilon) dx = \frac{1}{2} \frac{d}{dt} \left(\int_0^L \frac{v^2(x,t,\varepsilon)}{\sigma^2(x,\varepsilon)} dx \right) \tag{4.26}$$

and for the first integral on the right hand side

$$\begin{aligned} & \int_0^L v_{xx}(x,t,\varepsilon) v(x,t,\varepsilon) dx \\ = \; & \int_0^L \left[v_{xx}(x,t,\varepsilon) - \left(\tfrac{v(x,t,\varepsilon)}{x+\delta} \right)_x \right] v(x,t,\varepsilon) dx + \int_0^L \left(\tfrac{v(x,t,\varepsilon)}{x+\delta} \right)_x v(x,t,\varepsilon) dx \\ = \; & \left[v_x(x,t,\varepsilon) - \left(\tfrac{v(x,t,\varepsilon)}{x+\delta} \right) \right] v(x,t,\varepsilon) |_0^L - \int_0^L \left[v_x(x,t,\varepsilon) - \left(\tfrac{v(x,t,\varepsilon)}{x+\delta} \right) \right] v_x(x,t,\varepsilon) dx \\ & + \int_0^L \left(\tfrac{v(x,t,\varepsilon)}{x+\delta} \right)_x v(x,t,\varepsilon) dx. \end{aligned} \tag{4.27}$$

From (4.23) we have

$$\left[v_x(x,t,\varepsilon) - \left(\frac{v(x,t,\varepsilon)}{x+\delta} \right) \right]_{x=0} = 0$$

and

$$\left[v_x(x,t,\varepsilon) - \left(\frac{v(x,t,\varepsilon)}{x+\delta} \right) \right]_{x=L} = 0.$$

Then (4.27) becomes

$$\begin{aligned}
& \int_0^L v_{xx}(x,t,\varepsilon)v(x,t,\varepsilon)dx \\
= \quad & -\int_0^L \left[v_x(x,t,\varepsilon) - \left(\tfrac{v(x,t,\varepsilon)}{x+\delta}\right)\right] v_x(x,t,\varepsilon)dx + \int_0^L \left(\tfrac{v(x,t,\varepsilon)}{x+\delta}\right)_x v(x,t,\varepsilon)dx.
\end{aligned} \tag{4.28}$$

Using (4.26) and (4.28), (4.25) becomes

$$\begin{aligned}
& \tfrac{d}{dt}\left(\int_0^L \tfrac{v^2(x,t,\varepsilon)}{\sigma^2(x,\varepsilon)}dx\right) \\
= \quad & -\int_0^L \left[v_x(x,t,\varepsilon) - \left(\tfrac{v(x,t,\varepsilon)}{x+\delta}\right)\right] v_x(x,t,\varepsilon)dx + \int_0^L \left(\tfrac{v(x,t,\varepsilon)}{x+\delta}\right)_x v(x,t,\varepsilon)dx \\
& + \int_0^L \bar{\lambda}(x,\varepsilon)u_{xx}(x,t)v(x,t,\varepsilon)dx. \\
= \quad & -\int_0^L [v_x(x,t,\varepsilon)]^2 dx + 2\int_0^L \left[\tfrac{v(x,t,\varepsilon)v_x(x,t,\varepsilon)}{x+\delta}\right] dx - \int_0^L \tfrac{[v(x,t,\varepsilon)]^2}{(x+\delta)^2}dx \\
& + \int_0^L \bar{\lambda}(x,\varepsilon)u_{xx}(x,t)v(x,t,\varepsilon)dx.
\end{aligned} \tag{4.29}$$

Note that in (4.29), there exist $a > 0$ and $b > 0$ such that

$$2\int_0^L \left[\frac{v(x,t,\varepsilon)v_x(x,t,\varepsilon)}{x+\delta}\right] dx \le a\int_0^L \frac{[v(x,t,\varepsilon)]^2}{(x+\delta)^2}dx + \frac{1}{a}\int_0^L (v_x(x,t,\varepsilon))^2 dx \tag{4.30}$$

and

$$\int_0^L \bar{\lambda}(x,\varepsilon)u_{xx}(x,t)v(x,t,\varepsilon)dx \le \frac{b}{2}\int_0^L [\bar{\lambda}(x,\varepsilon)u_{xx}(x,t)]^2 dx + \frac{1}{2b}\int_0^L (v(x,t,\varepsilon))^2 dx. \tag{4.31}$$

Then from (4.29) we have

$$\begin{aligned}
\tfrac{d}{dt}\left(\int_0^L \tfrac{v^2(x,t,\varepsilon)}{\sigma^2(x,\varepsilon)}dx\right) \le \quad & -(1-\tfrac{1}{a})\int_0^L [v_x(x,t,\varepsilon)]^2 dx + (a-1)\int_0^L \tfrac{[v(x,t,\varepsilon)]^2}{(x+\delta)^2}dx \\
& + \tfrac{b}{2}\int_0^L [\bar{\lambda}(x,\varepsilon)u_{xx}(x,t)]^2 dx + \tfrac{1}{2b}\int_0^L [v(x,t,\varepsilon)]^2 dx.
\end{aligned} \tag{4.32}$$

Since $-\int_0^L \partial_{xx}dx$ has non-zero eigenvalues under Robin boundary conditions, using Poincare inequality we have

$$\int_0^L [v_x(x,t,\varepsilon)]^2 dx \ge \left(\frac{\pi}{L}\right)^2 \int_0^L [v(x,t,\varepsilon)]^2 dx \tag{4.33}$$

where $\left(\frac{\pi}{L}\right)^2$ is the first positive eigenvalue of $-\int_0^L \partial_{xx} dx$ with Robin conditions. Also note that

$$\int_0^L \frac{[v(x,t,\varepsilon)]^2}{(x+\delta)^2} dx \leq \int_0^L \frac{[v(x,t,\varepsilon)]^2}{\delta^2} dx = \frac{1}{\delta^2} \int_0^L [v(x,t,\varepsilon)]^2 dx \tag{4.34}$$

Then (4.32) becomes

$$\begin{aligned}
& \tfrac{d}{dt}\left(\int_0^L \tfrac{v^2(x,t,\varepsilon)}{\sigma^2(x,\varepsilon)} dx\right) \\
\leq\ & -\left(\tfrac{a-1}{a}\right)\left(\tfrac{\pi}{L}\right)^2 \int_0^L [v(x,t,\varepsilon)]^2 dx + \tfrac{(a-1)}{\delta^2} \int_0^L [v(x,t,\varepsilon)]^2 dx \\
& + \tfrac{b}{2} \int_0^L [\bar{\lambda}(x,\varepsilon) u_{xx}(x,t)]^2 dx + \tfrac{1}{2b} \int_0^L [v(x,t,\varepsilon)]^2 dx \\
=\ & -(a-1)\left[\tfrac{\pi^2}{aL^2} - \tfrac{1}{\delta} - \tfrac{1}{2b(a-1)}\right] \int_0^L [v(x,t,\varepsilon)]^2 dx \\
& + \tfrac{b}{2} \int_0^L [\bar{\lambda}(x,\varepsilon) u_{xx}(x,t)]^2 dx.
\end{aligned} \tag{4.35}$$

Assume that the modification only occurs on interval $[0, \eta_1(\varepsilon)]$ and interval $[\eta_2(\varepsilon), \eta_3(\varepsilon)]$ that include the two singular points. Since the modification $\sigma^2(x,\varepsilon)$ is always greater that $\sigma^2(x)$ over the two intervals, we have

$$[\bar{\lambda}(x,\varepsilon)]^2 = \left[1 - \frac{\sigma^2(x)}{\sigma^2(x,\varepsilon)}\right]^2 \leq 1$$

and

$$\frac{b}{2} \int_0^L [\bar{\lambda}(x,\varepsilon) u_{xx}(x,t)]^2 dx \leq \frac{b}{2} \left(\int_0^{\eta_1(\varepsilon)} + \int_{\eta_2(\varepsilon)}^{\eta_3(\varepsilon)}\right) [u_{xx}(x,t)]^2 dx. \tag{4.36}$$

Using Lemma 4.4.1, we have

$$\frac{b}{2} \int_0^L [\bar{\lambda}(x,\varepsilon) u_{xx}(x,t)]^2 dx \leq C(T) \left[\eta_1(\varepsilon) + \eta_3(\varepsilon) - \eta_2(\varepsilon)\right] \|u(x,0)\|_{H^2} \tag{4.37}$$

for some $C(T) > 0$. Let $K = (a-1)\left[\frac{\pi^2}{aL^2} - \frac{1}{\delta} - \frac{1}{2b(a-1)}\right]$ and $\eta(\varepsilon) = \left[\eta_1(\varepsilon) + \eta_3(\varepsilon) - \eta_2(\varepsilon)\right]$. If $\pi\delta > x^*$ and $x^* < L < \pi\delta$, then we can choose $a > 1$ such that $\frac{\pi^2}{aL^2} > \frac{1}{\delta^2}$. In this case, there exists

$b > 0$ such that $K > 0$. From (4.35) and (4.37) we have

$$\begin{aligned} & \tfrac{d}{dt}\left(\int_0^L \tfrac{v^2(x,t,\varepsilon)}{\sigma^2(x,\varepsilon)}dx\right) \\ \leq \; & -K\int_0^L \sigma^2(x,\varepsilon)\tfrac{v^2(x,t,\varepsilon)}{\sigma^2(x,\varepsilon)}dx + C(T)\eta(\varepsilon)\|u(x,0)\|_{H^2} \\ \leq \; & -K\varepsilon^2\int_0^L \tfrac{v^2(x,t,\varepsilon)}{\sigma^2(x,\varepsilon)}dx + C(T)\eta(\varepsilon)\|u(x,0)\|_{H^2}, \end{aligned} \tag{4.38}$$

assuming the minimum of $\sigma^2(x,\varepsilon)$ over $[0,L]$ is ε^2. With $v(x,0,\varepsilon) = 0$, (4.38) implies that

$$\begin{aligned} & \int_0^L \tfrac{v^2(x,t,\varepsilon)}{\sigma^2(x,\varepsilon)}dx \\ \leq \; & e^{-K\varepsilon^2 t}\left(\int_0^t e^{-K\varepsilon^2 s}ds\right)C(T)\eta(\varepsilon)\|u(x,0)\|_{H^2} \\ = \; & \tfrac{1-e^{-K\varepsilon^2 t}}{K\varepsilon^2}C(T)\eta(\varepsilon)\|u(x,0)\|_{H^2} \\ \leq \; & tC(T)\eta(\varepsilon)\|u(x,0)\|_{H^2}, \end{aligned} \tag{4.39}$$

which completes the proof of the theorem. □

In terms of $\rho(x,t,\varepsilon)$ and $\rho(x,t)$, we have

Corollary 4.4.3. *If $\pi\delta > x^*$ and $L \in (x^*, \pi\delta)$ then*

$$\int_0^L \frac{1}{(x+\delta)^2}\left[\rho(x,t,\varepsilon) - \frac{\sigma^2(x)}{\sigma^2(x,\varepsilon)}\rho(x,t)\right]^2 dx \leq tC(T)\|\rho(x,0)\|_{H^2}\eta(\varepsilon). \tag{4.40}$$

In Particular,

$$\left(\int_{\eta_1(\varepsilon)}^{\eta_2(\varepsilon)} + \int_{\eta_3(\varepsilon)}^{L}\right)[\rho(x,t,\varepsilon) - \rho(x,t)]^2 dx \leq (L+\delta)tC(T)\|\rho(x,0)\|_{H^2}\eta(\varepsilon) \tag{4.41}$$

Remark 4.4.1. In the proof above, if one applies integral by parts on the first term of (4.25), the boundary terms do not vanish in general and would cause difficulty in the calculation and estimation.

Without the conditions in Theorem 4.4.2, we have a partial result.

Lemma 4.4.4. *Let $\omega(x,t,\varepsilon) = \frac{v(x,t,\varepsilon)}{x+\delta}$. Then $\omega(x,t,\varepsilon)$ satisfies the homogenous Neumann boundary* *conditions.*

The Lemma can be easily verified from the previous results. Recall that eigenvalues of $-\partial_{xx}$ with Neumann boundary conditions are $\lambda_0 = 0, \lambda_n = \left(\frac{n\pi}{L}\right)^2 (n \geq 1)$ with eigenfunctions $\phi_0(x) = \frac{1}{\sqrt{L}}$ and $\phi_n(x) = \cos\frac{n\pi x}{\sqrt{L}}$. Let

$$\omega(x,t,\varepsilon) = \omega^0(t,\varepsilon)\phi_0(x) + \omega^{\perp}(x,t,\varepsilon),$$

where $\omega^0(t,\varepsilon) = \int_0^L \frac{1}{\sqrt{L}}\omega(x,t,\varepsilon)dx$. Then $\omega^{\perp}(x,t,\varepsilon) = \Sigma_{j=1}^{\infty} < \omega, \phi_j > \phi_j$. Now we define

$$v^0(x,t,\varepsilon) = (x+\delta)\omega^0 \text{ and } v^{\perp}(x,t,\varepsilon) = (x+\delta)\omega^{\perp}.$$

Then we have the follow theorem as a partial result.

Theorem 4.4.5. *For fixed $T > 0$, there exists $C(T,\varepsilon) > 0$ such that*

$$\int_0^L \frac{[v^{\perp}(x,t,\varepsilon)]^2}{\sigma^2(x,\varepsilon)}dx \leq \left[C(T,\varepsilon)\|u(x,t=0)\|_{H^2} + O(\varepsilon)\int_0^L \frac{1}{\sigma^2(x,\varepsilon)}dx\right]t. \tag{4.42}$$

Proof. With the assumption, we have

$$v(x,t,\varepsilon) = v^0(x,t,\varepsilon) + v^{\perp}(x,t,\varepsilon) \tag{4.43}$$

and (4.22) becomes

$$\begin{aligned} \partial_t v^0(x,t,\varepsilon) + \partial_t v^{\perp}(x,t,\varepsilon) = \quad & \tfrac{1}{2}\sigma^2(x,\varepsilon)\partial_{xx}\left[v^0(x,t,\varepsilon) + v^{\perp}(x,t,\varepsilon)\right] \\ & + \tfrac{1}{2}[\sigma^2(x,\varepsilon) - \sigma^2(x)]u_{xx}(x,t). \end{aligned} \tag{4.44}$$

Assume for the moment that

$$\partial_t v^0(x,t,\varepsilon) = O(\varepsilon)$$

for $t \in [0,T]$ with some $T > 0$. This will be verified later. Then we have

$$\partial_t v^{\perp}(x,t,\varepsilon) = \frac{1}{2}\sigma^2(x,\varepsilon)\partial_{xx}v^{\perp}(x,t,\varepsilon) + \frac{1}{2}[\sigma^2(x,\varepsilon) - \sigma^2(x)]u_{xx}(x,t) - O(\varepsilon)(t). \tag{4.45}$$

Using the same calculation in the proof of theorem 4.4.2, we have

$$\begin{aligned}
& \tfrac{d}{dt}\textstyle\int_0^L \frac{(v^{\perp}(x,t,\varepsilon))^2}{\sigma^2(x,\varepsilon)}dx \\
= \;& -\textstyle\int_0^L \left[v_x^{\perp}(x,t,\varepsilon)\right]^2 dx + 2\int_0^L \left[\frac{v^{\perp}(x,t,\varepsilon)v_x^{\perp}(x,t,\varepsilon)}{x+\delta}\right] dx - \int_0^L \frac{[v^{\perp}(x,t,\varepsilon)]^2}{(x+\delta)^2}dx \\
& + \textstyle\int_0^L \bar{\lambda}(x,\varepsilon)u_{xx}(x,t)v^{\perp}(x,t,\varepsilon)dx - O(\varepsilon)(t)\int_0^L \frac{1}{\sigma^2(x,\varepsilon)}dx. \\
= \;& -\textstyle\int_0^L (x+\delta)^2 \left[\frac{v_x^{\perp}(x,t,\varepsilon)}{x+\delta} - \frac{v^{\perp}(x,t,\varepsilon)}{(x+\delta)^2}\right]^2 dx + \int_0^L \bar{\lambda}(x,\varepsilon)u_{xx}(x,t)v^{\perp}(x,t,\varepsilon)dx \\
& -O(\varepsilon)\textstyle\int_0^L \frac{1}{\sigma^2(x,\varepsilon)}dx.
\end{aligned} \tag{4.46}$$

Each of these integrals in the above equation can be estimated as follows. For the first one, we have

$$\begin{aligned}
\textstyle\int_0^L (x+\delta)^2 \left[\frac{v_x^{\perp}(x,t,\varepsilon)}{x+\delta} - \frac{v^{\perp}(x,t,\varepsilon)}{(x+\delta)^2}\right]^2 dx \;& = \textstyle\int_0^L (x+\delta)^2 \left[\left(\frac{v^{\perp}(x,t,\varepsilon)}{x+\delta}\right)_x\right]^2 dx \\
& \geq \delta^2 \textstyle\int_0^L \left[\left(\frac{v^{\perp}(x,t,\varepsilon)}{x+\delta}\right)_x\right]^2 dx.
\end{aligned} \tag{4.47}$$

Note that $\omega(x)$ satisfies the Neumann boundary conditions. Given the definition of $v^{\perp}$, it can be verified that ω lies in the space spanned by $\phi_n(x) = \cos\frac{n\pi x}{L}$ $(n = 1,2,3,\cdots)$, which are the eigenfunctions of the boundary value problem

$$\omega''(x) = -\lambda\,\omega(x),\ \omega'(0) = \omega'(L) = 0$$

associated with eigenvalues $\left(\frac{\pi}{L}\right)^2, \left(\frac{2\pi}{L}\right)^2, \left(\frac{3\pi}{L}\right)^2, \cdots$. Therefore (4.48) gives

$$\int_0^L (x+\delta)^2 \left[\frac{v_x^{\perp}(x,t,\varepsilon)}{x+\delta} - \frac{v^{\perp}(x,t,\varepsilon)}{(x+\delta)^2}\right]^2 dx \geq \delta^2 \left(\frac{\pi}{L}\right)^2 \int_0^L \left(\frac{v^{\perp}(x,t,\varepsilon)}{x+\delta}\right)^2 dx, \tag{4.48}$$

which is equivalent to

$$-\int_0^L (x+\delta)^2 \left[\frac{v_x^{\perp}(x,t,\varepsilon)}{x+\delta} - \frac{v^{\perp}(x,t,\varepsilon)}{(x+\delta)^2}\right]^2 dx \leq -\delta^2 \left(\frac{\pi}{L}\right)^2 \int_0^L \left(\frac{v^{\perp}(x,t,\varepsilon)}{x+\delta}\right)^2 dx. \tag{4.49}$$

For the second integral on the right hand side of (4.47), we have

$$\begin{aligned}&\int_0^L \bar{\lambda}(x,\varepsilon)u_{xx}(x,t)v^{\perp}(x,t,\varepsilon)dx\\&=\int_0^L \left(\tfrac{1}{A}\bar{\lambda}(x,\varepsilon)(x+\delta)u_{xx}(x,t)\right)\left(A\tfrac{v^{\perp}(x,t,\varepsilon)}{x+\delta}\right)dx\\&\leq \tfrac{1}{2A^2}\int_0^L \bar{\lambda}^2(x,\varepsilon)(x+\delta)^2u_{xx}^2(x,t)dx+\tfrac{A^2}{2}\left(\tfrac{v^{\perp}(x,t,\varepsilon)}{x+\delta}\right)^2 dx,\end{aligned} \tag{4.50}$$

with some some positive constant A. Since we are looking at the finite period of time $[0,T]$, the integral that involves $u_{xx}(x,t)$ can be estimated using Lemma 4.4.1. Then we have

$$\int_0^L \bar{\lambda}^2(x,\varepsilon)(x+\delta)^2u_{xx}^2(x,t)dx \leq C(T)\|u(x,t=0)\|_{H^2}\int_0^L \bar{\lambda}^2(x,\varepsilon)(x+\delta)^2dx, \tag{4.51}$$

for some finite $C(T)>0$. Furthermore, let $(0,\eta_1)$ and $(x^*-\eta_2,x^*+\eta_2)$ be the modification intervals around the singular points where $\sigma^2(x,\varepsilon)-\sigma^2(x)>0$. Since the lengths of the intervals depend on the size of the modification, η_1 and η_2 are given in terms of ε. Hence we have

$$\begin{aligned}&\int_0^L \bar{\lambda}^2(x,\varepsilon)(x+\delta)^2u_{xx}^2(x,t)dx\\ \leq\ & C(T)\|u(x,t=0)\|_{H^2}\left(\int_0^{\eta_1(\varepsilon)}+\int_{x^*-\eta_2(\varepsilon)}^{x^*+\eta_2(\varepsilon)}\right)\bar{\lambda}^2(x,\varepsilon)(x+\delta)^2dx.\end{aligned} \tag{4.52}$$

With $\bar{\lambda}(x,\varepsilon)=1-\frac{\sigma^2(x)}{\sigma^2(x,\varepsilon)}$, one sees that $0<\bar{\lambda}(x,\varepsilon)<1$ and

$$\left(\int_0^{\eta_1(\varepsilon)}+\int_{x^*-\eta_2(\varepsilon)}^{x^*+\eta_2(\varepsilon)}\right)\bar{\lambda}^2(x,\varepsilon)(x+\delta)^2dx \leq \frac{1}{3}\left[(x+\delta)^3\Big|_0^{\eta_1(\varepsilon)}+(x+\delta)^3\Big|_{x^*-\eta_2(\varepsilon)}^{x^*+\eta_2(\varepsilon)}\right]. \tag{4.53}$$

From (4.53) and using the same notation for the constant, we have

$$\int_0^L \bar{\lambda}^2(x,\varepsilon)(x+\delta)^2u_{xx}^2(x,t)dx \leq C(T,\eta_1(\varepsilon),\eta_2(\varepsilon))\|u(x,t=0)\|_{H^2}. \tag{4.54}$$

Now using (4.47), (4.50), (4.51) and (4.55), we have obtained

$$
\begin{array}{rl}
 & \frac{d}{dt}\int_0^L \frac{(v^{\perp}(x,t,\varepsilon))^2}{\sigma^2(x,\varepsilon)}dx \\
\leq & -\left[\delta^2\left(\frac{\pi}{L}\right)^2-\frac{A^2}{2}\right]\int_0^L\left(\frac{v_x^{\perp}(x,t,\varepsilon)}{x+\delta}\right)^2dx+\frac{C(T,\eta_1(\varepsilon),\eta_2(\varepsilon))}{2A^2}\|u(x,t=0)\|_{H^2} \\
 & +O(\varepsilon)\int_0^L\frac{1}{\sigma^2(x,\varepsilon)}dx.
\end{array} \tag{4.55}
$$

Note that

$$
\begin{array}{rl}
\int_0^L\left(\frac{v_x^{\perp}(x,t,\varepsilon)}{x+\delta}\right)^2dx & =\int_0^L\frac{\sigma^2(x,\varepsilon)}{(x+\delta)^2}\frac{(v_x^{\perp}(x,t,\varepsilon))^2}{\sigma^2(x,\varepsilon)}dx \\
 & \geq\frac{\min_{[0,L]}\sigma^2(x,\varepsilon)}{(L+\delta)^2}\int_0^L\frac{(v_x^{\perp}(x,t,\varepsilon))^2}{\sigma^2(x,\varepsilon)}dx.
\end{array} \tag{4.56}
$$

For the simplicity and without loss of generality, we let $\min_{[0,L]}\sigma^2(x,\varepsilon)=\varepsilon^2$. We choose $A>0$ such that $\left(\frac{\pi}{L}\right)^2-\frac{A^2}{2\delta^2}>0$. Then (4.47) becomes

$$
\begin{array}{rl}
 & \frac{d}{dt}\int_0^L \frac{(v^{\perp}(x,t,\varepsilon))^2}{\sigma^2(x,\varepsilon)}dx \\
\leq & -\left[\left(\frac{\pi}{L}\right)^2-\frac{A^2}{2\delta^2}\right]\frac{\delta^2\varepsilon^2}{(L+\delta)^2}\int_0^L\frac{(v^{\perp}(x,t,\varepsilon))^2}{\sigma^2(x,\varepsilon)}dx+\frac{C(T,\eta_1(\varepsilon),\eta_2(\varepsilon))}{2A^2}\|u(x,t=0)\|_{H^2} \\
 & +O(\varepsilon)\int_0^L\frac{1}{\sigma^2(x,\varepsilon)}dx.
\end{array} \tag{4.57}
$$

Let

$$
B_1=\left[\left(\frac{\pi}{L}\right)^2-\frac{A^2}{2\delta^2}\right]\frac{\delta^2\varepsilon^2}{(L+\delta)^2}, \tag{4.58}
$$

$$
B_2=B_2(T,\varepsilon)=C(T,\varepsilon)\|u(x,t=0)\|_{H^2}+O(\varepsilon)\int_0^L\frac{1}{\sigma^2(x,\varepsilon)}dx, \tag{4.59}
$$

where $C(t,\varepsilon)=C(T,\eta_1(\varepsilon),\eta_2(\varepsilon))2A^2$. Let

$$
Y(t)=\int_0^L\frac{(v^{\perp}(x,t,\varepsilon))^2}{\sigma^2(x,\varepsilon)}dx.
$$

Then by solving the inequality

$$
Y_t(t)\leq -B_1Y(t)+B_2, \tag{4.60}
$$

we have

$$
Y(t)\leq e^{-B_1t}\int_0^t B_2(T,\varepsilon)e^{B_1s}ds. \tag{4.61}
$$

In terms of $v^{\perp}(x,t,\varepsilon)$, this is

$$\begin{aligned} \int_0^L \frac{(v^{\perp}(x,t,\varepsilon))^2}{\sigma^2(x,\varepsilon)}dx &\leq B_2(T,\varepsilon)e^{-\left[\left(\frac{\pi}{L}\right)^2-\frac{A^2}{2\delta^2}\right]\varepsilon^2 t}\int_0^t e^{\left[\left(\frac{\pi}{L}\right)^2-\frac{A^2}{2\delta^2}\right]\varepsilon^2 s}ds. \\ &\leq B_2(T,\varepsilon)t, \end{aligned} \tag{4.62}$$

where $B_2(T,\varepsilon)$ is given by (4.60) and the proof is complete. $\square$

Chapter 5

Estimation with further assumptions and improved boundary conditions

As we have seen in the previous sections, given the general form (4.4), the stationary solution (4.5) does not decay at the far right end and therefore is not a probability density function. In order to make the problem easier to handle and also for the results to be meaningful from practical point of view, we require additional assumptions and further improvement on the existing conditions. In this section, we increase the decay of $\frac{\mu(x,\varepsilon)}{\sigma^2(x,\varepsilon)}$ on the far right end so that certain property of the solutions to the modified equation (4.4) can be enhanced.

Consider the general form of the modified stationary solution

$$\rho(x,\infty,\varepsilon) = \frac{C_1 \int_0^x g(y,\varepsilon)dy}{g(x,\varepsilon)\sigma^2(x,\varepsilon)} \tag{5.1}$$

where

$$g(x,\varepsilon) = \exp\left(-\int_0^x \frac{2\mu(y,\varepsilon)}{\sigma^2(y,\varepsilon)}dy\right). \tag{5.2}$$

Let x^* be the second zero of $\sigma^2(x)$ and choose $x^* << X_{L_1} < X_{L_2} < \infty$. Let

$$\frac{\mu(x,\varepsilon)}{\sigma^2(x,\varepsilon)} = \begin{cases} \frac{1}{x+\delta},\ 0 \le x \le X_{L_1}, \\ \frac{x-X_{L_2}}{(X_{L_1}+\delta)(X_{L_1}-X_{L_2})},\ X_{L_1} < x \le X_{L_2}, \\ 0,\ X_{L_2} < x. \end{cases} \tag{5.3}$$

Then $\frac{\mu(x,\varepsilon)}{\sigma^2(x,\varepsilon)}$ is a piecewise continuous function that decays to zero at X_{L_2} and we have

$$\int_0^x \frac{2\mu(y,\varepsilon)}{\sigma^2(y,\varepsilon)}dy = \begin{cases} \ln\frac{(x+\delta)^2}{\delta^2},\ 0 \le x \le X_{L_1}, \\ \ln\frac{(X_{L_1}+\delta)^2}{\delta^2} + \frac{(x-X_{L_2})^2-(X_{L_1}-X_{L_2})^2}{(X_{L_1}+\delta)(X_{L_1}-X_{L_2})},\ X_{L_1} < x \le X_{L_2}, \\ \ln\frac{(X_{L_1}+\delta)^2}{\delta^2} + \frac{X_{L_2}-X_{L_1}}{(X_{L_1}+\delta)},\ X_{L_2} < x. \end{cases} \tag{5.4}$$

and

$$g(x,\varepsilon) = \begin{cases} \frac{\delta^2}{(x+\delta)^2},\ 0 \le x \le X_{L_1}, \\ \frac{\delta^2}{(X_{L_1}+\delta)^2} \cdot \exp\left[-\frac{(x-X_{L_2})^2-(X_{L_1}-X_{L_2})^2}{(X_{L_1}+\delta)(X_{L_1}-X_{L_2})}\right],\ X_{L_1} < x \le X_{L_2}, \\ \frac{\delta^2}{(X_{L_1}+\delta)^2} \cdot \exp\left[-\frac{X_{L_2}-X_{L_1}}{(X_{L_1}+\delta)}\right],\ X_{L_2} < x. \end{cases} \tag{5.5}$$

Then the stationary solution for the modified equation becomes

$$\rho(x,\infty,\varepsilon) = \begin{cases} \rho_1(x,\varepsilon),\ 0 \le x \le X_{L_1}, \\ \rho_2(x,\varepsilon),\ X_{L_1} < x \le X_{L_2}, \\ \rho_3(x,\varepsilon),\ X_{L_2} < x, \end{cases} \tag{5.6}$$

where

$$\rho_1(x,\varepsilon) = \frac{C_1 x(x+\delta)}{\delta\sigma^2(x,\varepsilon)}; \tag{5.7}$$

$$\rho_2(x,\varepsilon) = \frac{C_1 X_{L_1}(X_{L_1}+\delta)}{\delta \exp\left[-\frac{(x-X_{L_2})^2-(X_{L_1}-X_{L_2})^2}{(X_{L_1}+\delta)(X_{L_1}-X_{L_2})}\right] \cdot \sigma^2(x,\varepsilon)} + \frac{C_1 \int_{X_{L_1}}^x \exp\left[-\frac{(y-X_{L_2})^2}{(X_{L_1}+\delta)(X_{L_1}-X_{L_2})}\right] dy}{\exp\left[-\frac{(x-X_{L_2})^2}{(X_{L_1}+\delta)(X_{L_1}-X_{L_2})}\right] \cdot \sigma^2(x,\varepsilon)}; \tag{5.8}$$

$$\rho_3(x,\varepsilon) = \frac{C_1 X_{L_1}(X_{L_1}+\delta)}{\delta \exp\left[-\frac{X_{L_2}-X_{L_1}}{(X_{L_1}+\delta)}\right]\cdot \sigma^2(x,\varepsilon)} + \frac{C_1 \int_{X_{L_1}}^{X_{L_2}} \exp\left[-\frac{(y-X_{L_2})^2}{(X_{L_1}+\delta)(X_{L_1}-X_{L_2})}\right] dy}{\sigma^2(x,\varepsilon)} + \frac{C_1(x-X_{L_2})}{\sigma^2(x,\varepsilon)}; \tag{5.9}$$

For $L \geq X_{L_2}$, let

$$Q = \frac{X_{L_1}(X_{L_1}+\delta)}{\delta \exp\left[-\frac{X_{L_2}-X_{L_1}}{(X_{L_1}+\delta)}\right]} + \int_{X_{L_1}}^{X_{L_2}} \exp\left[-\frac{(y-X_{L_2})^2}{(X_{L_1}+\delta)(X_{L_1}-X_{L_2})}\right] dy - X_{L_2} \tag{5.10}$$

and let

$$\bar{\rho}(x,t) = \rho(x,t) - \frac{C_1 x}{\sigma^2(x)}\left(1+\frac{Q}{L}\right) \tag{5.11}$$

$$\bar{\rho}(x,t,\varepsilon) = \rho(x,t,\varepsilon) - \frac{C_1 x}{\sigma^2(x,\varepsilon)}\left(1+\frac{Q}{L}\right). \tag{5.12}$$

Then for $\beta < \frac{1}{2}$, we have

$$\bar{\rho}(x,t))\big|_{x=0} = \bar{\rho}(x,t))\big|_{x=L} = 0, \tag{5.13}$$

$$\bar{\rho}(x,t,\varepsilon))\big|_{x=0} = \bar{\rho}(x,t,\varepsilon))\big|_{x=L} = 0. \tag{5.14}$$

Given the above transformations, we can substitute $\rho(x,t) = \bar{\rho}(x,t) + \frac{C_1 x}{\sigma^2(x)}\left(1+\frac{Q}{L}\right)$ and $\rho(x,t,\varepsilon) = \bar{\rho}(x,t,\varepsilon) + \frac{C_1 x}{\sigma^2(x,\varepsilon)}\left(1+\frac{Q}{L}\right)$ into the Fokker-Planck Equation (4.4) and then we have

$$\begin{aligned} \bar{\rho}_t(x,t) &= \tfrac{1}{2}\left[\sigma^2(x)\bar{\rho}(x,t) + C_1\left(1+\tfrac{Q}{L}\right)x\right]_{xx} - \left[\mu(x)\bar{\rho}(x,t) + C_1\left(1+\tfrac{Q}{L}\right)\tfrac{\mu(x)}{\sigma^2(x)}x\right]_x \\ &= \tfrac{1}{2}\left[\sigma^2(x)\bar{\rho}(x,t)\right]_{xx} - \left[\mu(x)\bar{\rho}(x,t)\right]_x - \left[C_1\left(1+\tfrac{Q}{L}\right)\tfrac{\mu(x)}{\sigma^2(x)}x\right]_x, \end{aligned} \tag{5.15}$$

and similarly

$$\bar{\rho}_t(x,t,\varepsilon) = \frac{1}{2}\left[\sigma^2(x,\varepsilon)\bar{\rho}(x,t,\varepsilon)\right]_{xx} - \left[\mu(x,\varepsilon)\bar{\rho}(x,t,\varepsilon)\right]_x - \left[C_1\left(1+\frac{Q}{L}\right)\frac{\mu(x,\varepsilon)}{\sigma^2(x,\varepsilon)}x\right]_x. \tag{5.16}$$

Thus the difference of the modified and the non-modified equations can be described by the following equation

$$
\begin{aligned}
[\bar{\rho}(x,t,\varepsilon)-\bar{\rho}(x,t)]_t = \;& \tfrac{1}{2}\left[\sigma^2(x,\varepsilon)\bar{\rho}(x,t,\varepsilon)-\sigma^2(x,\varepsilon)\bar{\rho}(x,t)\right]_{xx} \\
& -[\mu(x,\varepsilon)\bar{\rho}(x,t,\varepsilon)-\mu(x,\varepsilon)\bar{\rho}(x,t)]_x \\
& -\left[C_1\left(1+\tfrac{Q}{L}\right)\left(\tfrac{\mu(x,\varepsilon)}{\sigma^2(x,\varepsilon)}-\tfrac{\mu(x)}{\sigma^2(x)}\right)x\right]_x .
\end{aligned} \tag{5.17}
$$

With the given relationship between μ and σ, we have $\frac{\mu(x,\varepsilon)}{\sigma^2(x,\varepsilon)} = \frac{1}{x+\delta} = \frac{\mu(x,\varepsilon)}{\sigma^2(x,\varepsilon)}$ and therefore the last term in the above equation is always zero. Let

$$
v(x,t,\varepsilon) = \bar{\rho}(x,t,\varepsilon)-\bar{\rho}(x,t), \tag{5.18}
$$

then the above equation (5.17) becomes

$$
\begin{aligned}
v_t(x,t,\varepsilon) = \;& \tfrac{1}{2}\left[(\sigma^2(x,\varepsilon)-\sigma^2(x))\bar{\rho}(x,t,\varepsilon)\right]_{xx}+\tfrac{1}{2}\left[\sigma^2(x,\varepsilon)v(x,t,\varepsilon)\right]_{xx} \\
& -[(\mu(x,\varepsilon)-\mu(x,\varepsilon))\bar{\rho}(x,t,\varepsilon)]_x-[\mu(x,\varepsilon)v(x,t,\varepsilon)]_x .
\end{aligned} \tag{5.19}
$$

From here we are trying to derive an equation that involves some measurement of v, such as $L-2$ norm or something like that in a sense, and hope this measurement of v can be estimated through the derived equation. With that in mind, we apply $\int_0^L \sigma^2(x)\cdot dx$ on each term of the above equation. First on the left hand side, we have

$$
\int_0^L \sigma^2(x)v(x,t,\varepsilon)v_t(x,t,\varepsilon)dx = \frac{1}{2}\left(\int_0^L \sigma^2(x)v^2(x,t,\varepsilon)dx\right)_t . \tag{5.20}
$$

For the first term on the right hand side, we have

$$
\begin{aligned}
& \textstyle\int_0^L \left[(\sigma^2(x,\varepsilon)-\sigma^2(x))\bar{\rho}(x,t,\varepsilon)\right]_{xx}\sigma^2(x)v(x,t,\varepsilon)dx \\
= \;& \left[(\sigma^2(x,\varepsilon)-\sigma^2(x))\bar{\rho}(x,t,\varepsilon)\right]_x\left[\sigma^2(x)v(x,t,\varepsilon)\right]\Big|_0^L \\
& -\textstyle\int_0^L \left[(\sigma^2(x,\varepsilon)-\sigma^2(x))\bar{\rho}(x,t,\varepsilon)\right]_x\left[\sigma^2(x)v(x,t,\varepsilon)\right]_x dx.
\end{aligned} \tag{5.21}
$$

Using the boundary conditions mentioned above, we can easily prove that

$$\left[(\sigma^2(x,\varepsilon)-\sigma^2(x))\bar{\rho}(x,t,\varepsilon)\right]_x\left[\sigma^2(x)v(x,t,\varepsilon)\right]\Big|_0^L=0. \tag{5.22}$$

Actually $\sigma^2(x,\varepsilon)=\sigma^2(x)$ for large x since the modification only applies in the small neighborhoods containing the sigular points while at $x=0^+$, we have $\sigma(0)=0$. Therefore the claim is true and for the above integral we have

$$\begin{aligned} & \int_0^L\left[(\sigma^2(x,\varepsilon)-\sigma^2(x))\bar{\rho}(x,t,\varepsilon)\right]_{xx}\sigma^2(x)v(x,t,\varepsilon)dx \\ = & -\int_0^L\left[(\sigma^2(x,\varepsilon)-\sigma^2(x))\bar{\rho}(x,t,\varepsilon)\right]_x\left[\sigma^2(x)v(x,t,\varepsilon)\right]_x dx. \end{aligned} \tag{5.23}$$

In the same way, for the other integrals on the right hand side, we have

$$\begin{aligned} & \int_0^L\left[\sigma^2(x)v(x,t,\varepsilon)\right]_{xx}\sigma^2(x)v(x,t,\varepsilon)dx \\ = & -\int_0^L\left[\sigma^2(x)v(x,t,\varepsilon)\right]_x^2 dx, \\ & \int_0^L[(\mu(x,\varepsilon)-\mu(x,\varepsilon))\bar{\rho}(x,t,\varepsilon)]_x\sigma^2(x)v(x,t,\varepsilon)dx \\ = & -\int_0^L[(\mu(x,\varepsilon)-\mu(x,\varepsilon))\bar{\rho}(x,t,\varepsilon)]\left[\sigma^2(x)v(x,t,\varepsilon)\right]_x dx, \end{aligned} \tag{5.24}$$

and

$$\begin{aligned} & \int_0^L[\mu(x,\varepsilon)v(x,t,\varepsilon)]_x\sigma^2(x)v(x,t,\varepsilon)dx \\ = & -\int_0^L[\mu(x,\varepsilon)v(x,t,\varepsilon)]\left[\sigma^2(x)v(x,t,\varepsilon)\right]_x dx. \end{aligned} \tag{5.25}$$

Therefore we have obtained

$$\begin{aligned} & \tfrac{1}{2}\left(\int_0^L\sigma^2(x)v^2(x,t,\varepsilon)dx\right)_t \\ = & -\tfrac{1}{2}\int_0^L\left[(\sigma^2(x,\varepsilon)-\sigma^2(x))\bar{\rho}(x,t,\varepsilon)\right]_x\left[\sigma^2(x)v(x,t,\varepsilon)\right]_x dx \\ & -\tfrac{1}{2}\int_0^L\left[\sigma^2(x)v(x,t,\varepsilon)\right]_x^2 dx \\ & +\int_0^L[(\mu(x,\varepsilon)-\mu(x,\varepsilon))\bar{\rho}(x,t,\varepsilon)]\left[\sigma^2(x)v(x,t,\varepsilon)\right]_x dx \\ & +\int_0^L[\mu(x,\varepsilon)v(x,t,\varepsilon)]\left[\sigma^2(x)v(x,t,\varepsilon)\right]_x dx. \end{aligned} \tag{5.26}$$

Now our goal is to find a upper bound for each of the integrals on the right hand side of the above equation so that we can estimate the left hand side in terms of t and ε. Given the relationship of $\sigma^2(x)$ and $\mu(x,\varepsilon)$, we can combine the first and the third integrals on the right hand side of the equation. Let

$$h(x,t,\varepsilon) = (\sigma^2(x,\varepsilon) - \sigma^2(x,\varepsilon))\bar{\rho}(x,t,\varepsilon) \tag{5.27}$$

then the sum of these two integrals gives

$$\begin{aligned}
& -\tfrac{1}{2}\int_0^L \left[h_x(x,t,\varepsilon) - \tfrac{2h(x,t,\varepsilon)}{x+\delta}\right]\left[\sigma^2(x)v(x,t,\varepsilon)\right]_x dx \\
= \;& -\tfrac{1}{2}\int_0^L \left[\tfrac{h_x(x,t,\varepsilon)}{(x+\delta)^2} - \tfrac{2h(x,t,\varepsilon)}{(x+\delta)^3}\right](x+\delta)^2\left[\sigma^2(x)v(x,t,\varepsilon)\right]_x dx \\
= \;& -\tfrac{1}{2}\int_0^L \left[\tfrac{h(x,t,\varepsilon)}{(x+\delta)^2}\right]_x (x+\delta)^2\left[\sigma^2(x)v(x,t,\varepsilon)\right]_x dx.
\end{aligned} \tag{5.28}$$

For the last integral, using integral by part, we have

$$\begin{aligned}
& \int_0^L [\mu(x,\varepsilon)v(x,t,\varepsilon)]\left[\sigma^2(x)v(x,t,\varepsilon)\right]_x dx \\
= \;& \int_0^L \tfrac{\left[\sigma^2(x)v(x,t,\varepsilon)\right]}{x+\delta}\left[\sigma^2(x)v(x,t,\varepsilon)\right]_x dx \\
= \;& \tfrac{1}{2}\int_0^L \tfrac{1}{(x+\delta)^2}\left[\sigma^2(x)v(x,t,\varepsilon)\right]^2 dx
\end{aligned} \tag{5.29}$$

since $\left[\sigma^2(x)v(x,t,\varepsilon)\right]^2\Big|_0^L = 0$ with the given boundary conditions applied. Now let

$$w(x,t,\varepsilon) = \left[\frac{h(x,t,\varepsilon)}{(x+\delta)^2}\right]_x (x+\delta)^2, \tag{5.30}$$

then we have

$$\begin{aligned}
& \tfrac{d}{dt}\left(\int_0^L \sigma^2(x)v^2(x,t,\varepsilon)dx\right) \\
= \;& -\int_0^L w(x,t,\varepsilon)\left[\sigma^2(x)v(x,t,\varepsilon)\right]_x dx - \int_0^L \left[\sigma^2(x)v(x,t,\varepsilon)\right]_x^2 dx \\
& + \int_0^L \tfrac{1}{(x+\delta)^2}\left[\sigma^2(x)v(x,t,\varepsilon)\right]^2 dx.
\end{aligned} \tag{5.31}$$

Note that

$$\begin{array}{rl} & \left| -\int_0^L w(x,t,\varepsilon)\left[\sigma^2(x)v(x,t,\varepsilon)\right]_x dx \right| \\ = & \left| -\int_0^L (kw(x,t,\varepsilon))\left(\frac{1}{k}\left[\sigma^2(x)v(x,t,\varepsilon)\right]_x\right) dx \right| \\ \leq & \left(\int_0^L k^2 w^2(x,t,\varepsilon)dx\right)^{\frac{1}{2}} \left(\int_0^L \frac{1}{k^2}\left[\sigma^2(x)v(x,t,\varepsilon)\right]_x^2 dx\right)^{\frac{1}{2}} \\ \leq & \frac{k^2}{2}\int_0^L w^2(x,t,\varepsilon)dx + \frac{1}{2k^2}\int_0^L \left[\sigma^2(x)v(x,t,\varepsilon)\right]_x^2 dx, \end{array} \tag{5.32}$$

for some positive number k and also note that

$$\frac{1}{(x+\delta)^2} \leq \frac{1}{\delta^2}. \tag{5.33}$$

Then the RHS can be further estimated and organized as follows.

$$\begin{array}{rl} & \frac{d}{dt}\left(\int_0^L \sigma^2(x)v^2(x,t,\varepsilon)dx\right) \\ \leq & \frac{k^2}{2}\int_0^L w^2(x,t,\varepsilon)dx - \left(1-\frac{1}{2k^2}\right)\int_0^L \left[\sigma^2(x)v(x,t,\varepsilon)\right]_x^2 dx, \\ & +\frac{1}{\delta^2}\int_0^L \left[\sigma^2(x)v(x,t,\varepsilon)\right]^2 dx. \end{array} \tag{5.34}$$

Here we can choose k such that $\left(1-\frac{1}{2k^2}\right) > 0$.

Lemma 5.0.6. *Given the above boundary conditions and the assumption on* μ *and* σ, *the following inequality holds*

$$\int_0^L \left[\sigma^2(x)v(x,t,\varepsilon)\right]^2 dx \leq \frac{L^2}{2}\int_0^L \left[\sigma^2(x)v(x,t,\varepsilon)\right]_x^2 dx \tag{5.35}$$

Proof. Note that

$$\sigma^2(x)v(x,t,\varepsilon) = \int_0^x \left[\sigma^2(x)v(x,t,\varepsilon)\right]_x \cdot 1 \cdot dx \tag{5.36}$$

Then we have

$$|\sigma^2(x)v(x,t,\varepsilon)| \leq \left(\int_0^x \left[\sigma^2(x)v(x,t,\varepsilon)\right]_x^2 dx\right)^{\frac{1}{2}} \left(\int_0^x \cdot 1^2 \cdot dx\right)^{\frac{1}{2}} \tag{5.37}$$

and thus

$$\begin{aligned} \left(\sigma^2(x)v(x,t,\varepsilon)\right)^2 & \le x\int_0^x \left[\sigma^2(x)v(x,t,\varepsilon)\right]_x^2 dx \\ & \le x\int_0^L \left[\sigma^2(x)v(x,t,\varepsilon)\right]_x^2 dx. \end{aligned} \tag{5.38}$$

Taking the integral from 0 to L and then we have completed the proof. $\square$

Therefore we have

$$\begin{aligned} & \tfrac{d}{dt}\left(\int_0^L \sigma^2(x)v^2(x,t,\varepsilon)dx\right) \\ \le\ & \tfrac{k^2}{2}\int_0^L w^2(x,t,\varepsilon)dx - \left(1-\tfrac{1}{2k^2}\right)\tfrac{2}{L^2}\int_0^L \left[\sigma^2(x)v(x,t,\varepsilon)\right]^2 dx, \\ & +\tfrac{1}{\delta^2}\int_0^L \left[\sigma^2(x)v(x,t,\varepsilon)\right]^2 dx. \\ =\ & \tfrac{k^2}{2}\int_0^L w^2(x,t,\varepsilon)dx - \left(\tfrac{2}{L^2}-\tfrac{1}{L^2k^2}-\tfrac{1}{\delta^2}\right)\int_0^L \left[\sigma^2(x)v(x,t,\varepsilon)\right]^2 dx. \end{aligned} \tag{5.39}$$

Let

$$\tilde{G}(t,\varepsilon,k) = \frac{k^2}{2}\int_0^L w^2(x,t,\varepsilon)dx$$

and

$$\tilde{K} = \frac{2}{L^2} - \frac{1}{L^2k^2} - \frac{1}{\delta^2}.$$

Then we have

$$\frac{d}{dt}\left(\int_0^L \sigma^2(x)v^2(x,t,\varepsilon)dx\right) \le \tilde{G}(t,\varepsilon,k) - \tilde{K}\int_0^L \left[\sigma^2(x)v(x,t,\varepsilon)\right]^2 dx. \tag{5.40}$$

Note that

$$\begin{aligned} & \int_0^L \sigma^2(x)v^2(x,t,\varepsilon)dx \\ \ge\ & \int_0^L \tfrac{1}{\sigma^2(x,\varepsilon)}\left(\sigma^2(x)v(x,t,\varepsilon)\right)^2 dx \\ \ge\ & \tfrac{1}{\max_{[0,L]}\sigma^2(x,\varepsilon)}\int_0^L \left(\sigma^2(x)v(x,t,\varepsilon)\right)^2 dx, \end{aligned} \tag{5.41}$$

which gives

$$\int_0^L \left(\sigma^2(x)v(x,t,\varepsilon)\right)^2 \le \max_{[0,L]}\sigma^2(x,\varepsilon)\int_0^L \sigma^2(x)v^2(x,t,\varepsilon)dx. \tag{5.42}$$

Now we have

$$\frac{d}{dt}\left(\int_0^L \left(\sigma^2(x)v(x,t,\varepsilon)\right)^2\right) \le \max_{[0,L]} \sigma^2(x,\varepsilon)\frac{d}{dt}\left(\int_0^L \sigma^2(x)v^2(x,t,\varepsilon)dx\right). \qquad (5.43)$$

From inequality (5.40), we have

$$\begin{aligned} & \tfrac{d}{dt}\left(\int_0^L \left(\sigma^2(x)v(x,t,\varepsilon)\right)^2\right) \\ \le\;& \max_{[0,L]}\sigma^2(x,\varepsilon)\tilde{G}(t,\varepsilon,k) - \tilde{K}\max_{[0,L]}\sigma^2(x,\varepsilon)\int_0^L \left[\sigma^2(x)v(x,t,\varepsilon)\right]^2 dx. \end{aligned} \qquad (5.44)$$

Let $G = \max_{[0,L]}\sigma^2(x,\varepsilon)\tilde{G}(t,\varepsilon,k)$ and $K = \tilde{K}\max_{[0,L]}\sigma^2(x,\varepsilon)$. By solving this inequality, we have

$$\int_0^L \left(\sigma^2(x)v(x,t,\varepsilon)\right)^2 dx \le \exp(-Kt)\int_0^t \exp(Ks)G(s,\varepsilon,k)ds, \qquad (5.45)$$

where

$$\begin{aligned} G(t,\varepsilon,k) &= \tfrac{k^2}{2}\max_{[0,L]}\sigma^2(x,\varepsilon)\int_0^L w^2(x,t,\varepsilon)dx \\ &= \tfrac{k^2}{2}\max_{[0,L]}\sigma^2(x,\varepsilon)\int_0^L \left(\left[\tfrac{(\sigma^2(x,\varepsilon)-\sigma^2(x))\bar{\rho}(x,t,\varepsilon)}{(x+\delta)^2}\right]_x (x+\delta)^2\right)^2 dx. \end{aligned} \qquad (5.46)$$

depends on the $\sigma(x,\varepsilon)-\sigma^2(x)$, which is controlled by ε. Meanwhile, we can require $K>0$ by choosing appropriate L and k.

Now we want to estimate $G(t,\varepsilon,k)$. Using (5.12), we have

$$\begin{aligned} & \int_0^L \left(\left[\tfrac{(\sigma^2(x,\varepsilon)-\sigma^2(x))\bar{\rho}(x,t,\varepsilon)}{(x+\delta)^2}\right]_x (x+\delta)^2\right)^2 dx \\ =\;& \int_0^L \left(\left[\tfrac{(\sigma^2(x,\varepsilon)-\sigma^2(x))\rho(x,t,\varepsilon)}{(x+\delta)^2}\right]_x (x+\delta)^2\right)^2 dx \\ & -C_1\left(1+\tfrac{Q}{L}\right)\int_0^L \left(\left[\tfrac{x(\sigma^2(x,\varepsilon)-\sigma^2(x))}{\sigma^2(x,\varepsilon)(x+\delta)^2}\right]_x (x+\delta)^2\right)^2 dx. \end{aligned} \qquad (5.47)$$

On the right hand side of the above equation, the second term is independent of time t and for the first term, we have

$$
\begin{aligned}
&\int_0^L \left(\left[\frac{(\sigma^2(x,\varepsilon)-\sigma^2(x))\rho(x,t,\varepsilon)}{(x+\delta)^2} \right]_x (x+\delta)^2 \right)^2 dx \\
= \; &\int_0^L \left(\left[\frac{(\sigma^2(x,\varepsilon)-\sigma^2(x))\rho(x,\infty,\varepsilon)}{(x+\delta)^2} \right]_x (x+\delta)^2 \right)^2 dx \\
&+\int_0^L \left(\left[\frac{(\sigma^2(x,\varepsilon)-\sigma^2(x))(\rho(x,t,\varepsilon)-\rho(x,\infty,\varepsilon))}{(x+\delta)^2} \right]_x (x+\delta)^2 \right)^2 dx,
\end{aligned} \tag{5.48}
$$

which can be estimated numerically since $\rho(x,\infty,\varepsilon)$ is known.

Chapter 6

Possible Modifications with Numerics

In this chapter, we study three types of possible modifications on term $\sigma^2(x)$. The idea is that the singularity of the SDE and the corresponding PDE can be overcome or avoided by slightly modifying $\sigma^2(x)$ only near its zeros $x = 0$ and $x = C_0^{\frac{1}{1-\beta}} (= x^*)$, meanwhile the modified equations can somewhat capture the dynamics of the original equation. Since the stationary solution of (4.4) exists as long as $\sigma(x, \varepsilon) \neq 0$, we are interested in the blow-up speed of the stationary solution at the two singular points as the modified equation approaches the original or the non-modified equation.

We first look at a simple and straightforward modification where the zeros are "cut off" by a piece-wise function. In the second modification, we improve the the "cut-off" at the the first zero by extending the $\sigma^2(x)$ along the tangent line near $x = 0$. For the third type of modification, we add a smooth function to $\sigma^2(x)$ so that the zeros are "covered up" while the other points away from the two zeros are almost unchanged. It needs to be pointed out that, due to the differentiability, equation (4.4) is no longer meaningful under the first two types of modification. However, we can apply the these types of modifications to the transformed equation (4.14) and then return certain results we obtain from studying (4.14) to equation (4.4) through the transformation (4.12). Compared with the first two modifications through numerical experiments, the asymptotic behavior of the solution at the singular points under the third modification appears relatively better.

6.1 A Simple Modification

6.1.1 "Cutoff" at the Singular Points

In this section, we want to analyze the dynamics of the stationary solution at the singular points. We first look at a very simple and straightforward modification of function $\sigma(x)$, regardless of the FPEs. A modification is given as follows.

$$\sigma^2(\varepsilon_1,\varepsilon_2,x) = \begin{cases} \varepsilon_1^2, & \text{for } 0 \le x \le s_1(\varepsilon_1), \\ \sigma^2(x), & \text{for } s_1(\varepsilon_1) < x \le s_2(\varepsilon_2), \\ \varepsilon_2^2, & \text{for } s_2(\varepsilon_2) < x \le s_3(\varepsilon_2), \\ \sigma^2(x), & \text{for } s_3(\varepsilon_2) < x \le 1, \end{cases} \tag{6.1}$$

where $s_1(\varepsilon_1) > 0$ is the first solution of equation $\sigma^2(x) = \varepsilon_1^2$ and $s_2(\varepsilon_2) > 0$ and $s_3(\varepsilon_2) > 0$ are the second and the third solutions of equation $\sigma^2(x) = \varepsilon_2^2$ that satisfy $0 < s_1(\varepsilon_1) < s_2(\varepsilon_2) < s_3(\varepsilon_2) < 1$. (See Figure 6.1) In this case, we have the following results:

Lemma 6.1.1. *For $C_0 > 0$ and sufficiently small ε_1 and ε_2, $s_1(\varepsilon_1)$ can be expressed in terms of ε_1 as follows*

$$s_1(\varepsilon_1) = \left(\frac{\varepsilon_1}{C_0}\right)^{\frac{1}{\beta}} \left[1 + \frac{1}{\beta C_0^{\frac{1}{\beta}-1}}\varepsilon_1^{\frac{1}{\beta}-1} + \frac{3-\beta}{2\beta^2 C_0^{\frac{2}{\beta}-2}}\varepsilon_1^{\frac{2}{\beta}-2} + o\left(\varepsilon_1^{\frac{2}{\beta}-2}\right)\right], \tag{6.2}$$

$s_2(\varepsilon_2)$ and $s_3(\varepsilon_2)$ can be expressed in terms of ε_2 as follows

$$s_2(\varepsilon_2) = C_0^{\frac{1}{1-\beta}} - \frac{1}{1-\beta}\varepsilon_2 - \frac{\beta}{(1-\beta)^2 C_0^{\frac{1}{1-\beta}}}\varepsilon_2^2 + o(\varepsilon_2^2), \tag{6.3}$$

$$s_3(\varepsilon_2) = C_0^{\frac{1}{1-\beta}} + \frac{1}{1-\beta}\varepsilon_2 + \frac{\beta}{(1-\beta)^2 C_0^{\frac{1}{1-\beta}}}\varepsilon_2^2 + o(\varepsilon_2^2). \tag{6.4}$$

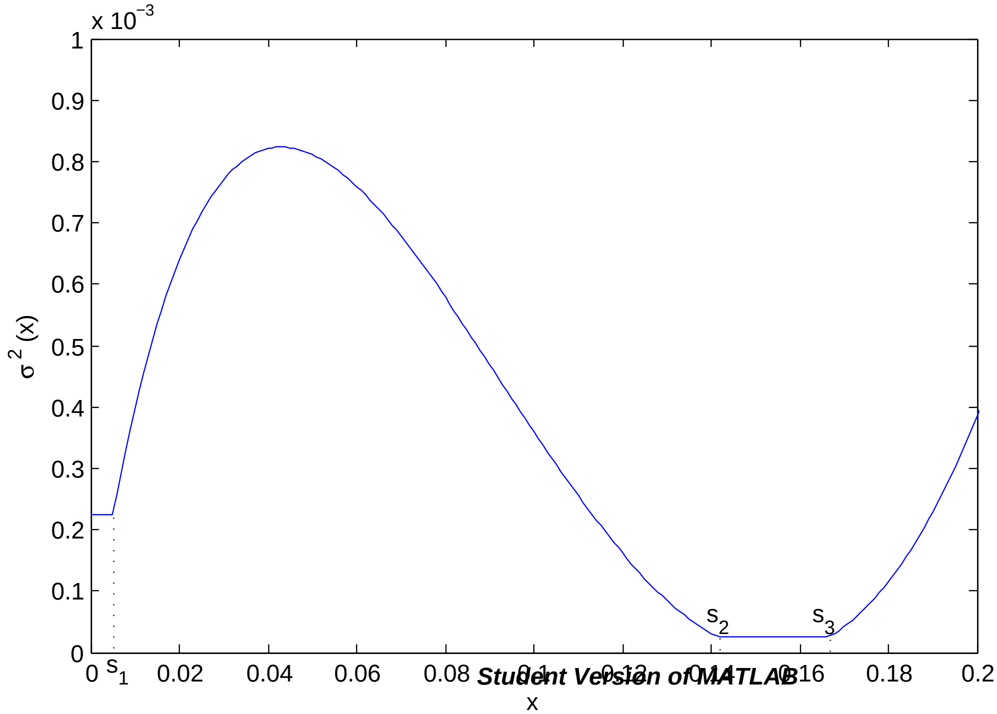

Figure 6.1: First Modification of $\sigma(x)$.

Proof. Note that, for $\varepsilon_1 > 0$ small enough, $C_0 s_1^{\beta}(\varepsilon_1) > s_1(\varepsilon_1)$ and equation

$$(s_1(\varepsilon_1) - C_0 s_1^{\beta}(\varepsilon_1))^2 = \varepsilon_1^2 \tag{6.5}$$

is equivalent to

$$C_0 s_1^{\beta}(\varepsilon_1) - s_1(\varepsilon_1) = \varepsilon_1, \tag{6.6}$$

which implies

$$\lim_{\varepsilon_1 \to 0^+} \frac{s_1^{\beta}(\varepsilon_1)}{\varepsilon_1} = \lim_{\varepsilon_1 \to 0^+} \frac{1}{C_0 - s^{1-\beta}(\varepsilon_1)} = \frac{1}{C_0}. \tag{6.7}$$

So we have

$$s_1(\varepsilon_1) = \left(\frac{\varepsilon_1}{C_0}\right)^{\frac{1}{\beta}} + o\left(\varepsilon_1^{\frac{1}{\beta}}\right). \tag{6.8}$$

Write

$$s_1(\varepsilon_1) = \left(\frac{\varepsilon_1}{C_0}\right)^{\frac{1}{\beta}} [1 + a(\varepsilon_1)], \tag{6.9}$$

where $a(\cdot)$ is to be determined and then we have

$$\frac{\varepsilon_1}{C_0}[1+a(\varepsilon_1)]^{\beta} - \left(\frac{\varepsilon_1}{C_0}\right)^{\frac{1}{\beta}}[1+a(\varepsilon_1)] = \varepsilon_1, \tag{6.10}$$

which leads to

$$\lim_{\varepsilon_1\to 0^+}\frac{[1+a(\varepsilon_1)]^{\beta}-C_0}{\varepsilon_1^{\frac{1}{\beta}-1}} = \lim_{\varepsilon_1\to 0^+}\frac{1+a(\varepsilon_1)}{C_0^{\frac{1}{\beta}-1}}. \tag{6.11}$$

Using I'Hopital's rule on the left hand side, we have

$$\lim_{\varepsilon_1\to 0^+}\frac{\beta[1+a(\varepsilon_1)]^{\beta-1}a'(\varepsilon_1)}{\left(\frac{1}{\beta}-1\right)\varepsilon_1^{\frac{1}{\beta}-2}} = \lim_{\varepsilon_1\to 0^+}\frac{\beta[1+a(\varepsilon_1)]^{\beta-1}}{\frac{1}{\beta}-1}\cdot\lim_{\varepsilon_1\to 0^+}\frac{a'(\varepsilon_1)}{\varepsilon_1^{\frac{1}{\beta}-2}} = \frac{1}{C_0^{\frac{1}{\beta}-1}}, \tag{6.12}$$

which implies

$$a'(\varepsilon_1) = \frac{1-\beta}{\beta^2 C_0^{\frac{1}{\beta}-1}}\varepsilon_1^{\frac{1}{\beta}-2} + o(\varepsilon_1^{\frac{1}{\beta}-2}) \tag{6.13}$$

and

$$a(\varepsilon_1) = \frac{1}{\beta C_0^{\frac{1}{\beta}-1}}\varepsilon_1^{\frac{1}{\beta}-1} + o(\varepsilon_1^{\frac{1}{\beta}-1}). \tag{6.14}$$

Thus we have

$$s_1(\varepsilon_1) = \frac{1}{C_0^{\frac{1}{\beta}}}\varepsilon_1^{\frac{1}{\beta}} + \frac{1}{\beta C_0^{\frac{2}{\beta}-1}}\varepsilon_1^{\frac{2}{\beta}-1} + o(\varepsilon_1^{\frac{2}{\beta}-1}). \tag{6.15}$$

Similarly let

$$s_1(\varepsilon_1) = \left(\frac{\varepsilon_1}{C_0}\right)^{\frac{1}{\beta}}\left[1+\frac{1}{\beta C_0^{\frac{1}{\beta}-1}}\varepsilon_1^{\frac{1}{\beta}-1} + k\varepsilon_1^{\frac{1}{\beta}-1+\alpha}\right], \tag{6.16}$$

where k and $\alpha > 0$ are to be determined and then we can obtain

$$\lim_{\varepsilon_1\to 0}\frac{\beta k\alpha(\frac{1}{\beta}-1+\alpha)\varepsilon_1^{\alpha-1}}{2\left(\frac{1}{\beta}-1\right)^2\varepsilon_1^{\frac{1}{\beta}-2}} = \frac{3-\beta}{2\beta C_0^{\frac{2}{\beta}-2}}, \tag{6.17}$$

which implies $\alpha = \frac{1}{\beta} - 1$. Then we can solve for k and have

$$k = \frac{3-\beta}{2\beta^2 C_0^{\frac{2}{\beta}-2}}. \tag{6.18}$$

Therefore we have

$$s_1(\varepsilon_1) = \left(\frac{\varepsilon_1}{C_0}\right)^{\frac{1}{\beta}} \left[1 + \frac{1}{\beta C_0^{\frac{1}{\beta}-1}} \varepsilon_1^{\frac{1}{\beta}-1} + \frac{3-\beta}{2\beta^2 C_0^{\frac{2}{\beta}-2}} \varepsilon_1^{\frac{2}{\beta}-2} + o\left(\varepsilon_1^{\frac{2}{\beta}-2}\right)\right]. \tag{6.19}$$

For $0 < \beta < 1$, we have $\frac{2}{\beta} - 1 > 1$ and thus

$$s_1(\varepsilon_1) = \frac{1}{C_0^{\frac{1}{\beta}}} \varepsilon_1^{\frac{1}{\beta}} + \frac{1}{\beta C_0^{\frac{2-\beta}{\beta}}} \varepsilon_1^{\frac{2-\beta}{\beta}} + o\left(\varepsilon_1^{\frac{2-\beta}{\beta}}\right). \tag{6.20}$$

For $C_0 > 0$, equation $s_2(\varepsilon_2) - C_0 s_2^{\beta}(\varepsilon_2) = -\varepsilon_2$ implies

$$\begin{aligned}
s_2(0) &= C_0^{\frac{1}{1-\beta}} \\
s_2'(0) &= \tfrac{1}{C_0 \beta s_2^{\beta-1}(0) - 1} \\
&= -\tfrac{1}{1-\beta} \\
s_2''(0) &= -\tfrac{C_0 \beta (\beta-1) s_2^{\beta-2}(0) s_2'(0)}{\left(C_0 \beta s_2^{\beta-1}(0) - 1\right)^2} \\
&= -\tfrac{\beta}{(1-\beta)^2 C_0^{\frac{1}{1-\beta}}}
\end{aligned} \tag{6.21}$$

then we have

$$s_2(\varepsilon_2) = C_0^{\frac{1}{1-\beta}} - \frac{1}{1-\beta}\varepsilon_2 - \frac{\beta}{(1-\beta)^2 C_0^{\frac{1}{1-\beta}}} \varepsilon_2^2 + o(\varepsilon_2^2) \tag{6.22}$$

and in the same way, we have

$$s_3(\varepsilon_2) = C_0^{\frac{1}{1-\beta}} + \frac{1}{1-\beta}\varepsilon_2 + \frac{\beta}{(1-\beta)^2 C_0^{\frac{1}{1-\beta}}} \varepsilon_2^2 + o(\varepsilon_2^2). \tag{6.23}$$

□

6.1.2 Stationary Solution of the FPE using "Cut-off" Modification

With the given cutoff points $s_1(\varepsilon_1)$, $s_2(\varepsilon_2)$ and $s_3(\varepsilon_2)$, we can find the constant in the general form of the stationary solution in terms of ε_1 and ε_2 so that the asymptotic behavior of the stationary solution can be described in terms of the parameters. In this section, we carry out the calculation and estimate the speed or order of the stationary solution $p(x,\infty,\varepsilon_1,\varepsilon_2)$ approaching ∞ as $(\varepsilon_1,\varepsilon_2)\to(0,0)$, given the above modification. For simplicity, we consider

$$\begin{aligned}
&\int_0^1 \frac{x^2}{\sigma^2(\varepsilon_1,\varepsilon_2,x)}dx \\
=\ & \left(\int_0^{s_1(\varepsilon_1)}+\int_{s_1(\varepsilon_1)}^{s_2(\varepsilon_2)}+\int_{s_2(\varepsilon_2)}^{s_3(\varepsilon_2)}+\int_{s_3(\varepsilon_2)}^{1}\right)\frac{x^2}{\sigma^2(\varepsilon_1,\varepsilon_2,x)}dx \\
=\ & I_1+I_2+I_3+I_4.
\end{aligned} \tag{6.24}$$

Then we have

$$\begin{aligned}
I_1 &= \frac{1}{3\varepsilon_1^2}\left[s_1^3(\varepsilon_1)-0\right] \\
&= \frac{\varepsilon_1^{\frac{3}{\beta}-2}}{3C_0^{\frac{1}{\beta}}}\left[1+\frac{1}{\beta C_0^{\frac{1}{\beta}-1}}\varepsilon_1^{\frac{1}{\beta}-1}+\frac{3-\beta}{2\beta^2C_0^{\frac{2}{\beta}-2}}\varepsilon_1^{\frac{2}{\beta}-2}+o\left(\varepsilon_1^{\frac{2}{\beta}-2}\right)\right]^3 \\
I_3 &= \frac{1}{3\varepsilon_2^2}\left[s_3^3(\varepsilon_2)-s_2^3(\varepsilon_2)\right] \\
&= \frac{2}{(1-\beta)\varepsilon_2}\left(1+(\Delta-1)C_0^{\frac{1}{1-\beta}}-\Delta C_0^{\frac{2}{1-\beta}}\right)\left[1+\frac{\beta\varepsilon_2}{(1-\beta)C_0^{\frac{1}{1-\beta}}}+o(\varepsilon_2)\right].
\end{aligned} \tag{6.25}$$

For I_2 and I_4, we have the following calculation. Let $y=x^{1-\beta}-C_0$ then we have

$$x=(y+C_0)^{\frac{1}{1-\beta}},\ dx=\frac{1}{1-\beta}(y+C_0)^{\frac{\beta}{1-\beta}}dy$$

$$x=s_1(\varepsilon_1)\Leftrightarrow y=s_1^{1-\beta}(\varepsilon_1)-C_0,$$

$$x=s_2(\varepsilon_2)\Leftrightarrow y=s_2^{1-\beta}(\varepsilon_2)-C_0.$$

Thus

$$\begin{aligned} I_2 &= \int_{s_1(\varepsilon_1)}^{s_2(\varepsilon_2)} \frac{x^2}{(x-C_0x^\beta)^2} dx \\ &= \int_{s_1(\varepsilon_1)}^{s_2(\varepsilon_2)} \frac{x^{2(1-\beta)}}{(x^{1-\beta}-C_0)^2} dx \\ &= \int_{s_1^{1-\beta}(\varepsilon_1)-C_0}^{s_2^{1-\beta}(\varepsilon_2)-C_0} \frac{(y+C_0)^{\frac{2}{1-\beta}}}{y^2} \frac{1}{1-\beta} (y+C_0)^{\frac{\beta}{1-\beta}} dy \\ &= \frac{1}{1-\beta} \int_{s_1^{1-\beta}(\varepsilon_1)}^{s_2^{1-\beta}(\varepsilon_2)} (y+C_0)^{\frac{2+\beta}{1-\beta}} \frac{1}{y^2} dy. \end{aligned} \tag{6.26}$$

Note that

$$(y+C_0)^{\frac{2+\beta}{1-\beta}} = C_0^{\frac{2+\beta}{1-\beta}} + \frac{2+\beta}{1-\beta} C_0^{\frac{1+2\beta}{1-\beta}} y + \left(\frac{2+\beta}{1-\beta}\right)\left(\frac{1+2\beta}{1-\beta}\right) C_0^{\frac{3\beta}{1-\beta}} y^2 + o(y^2). \tag{6.27}$$

Therefore we have

$$I_2 = \frac{1}{1-\beta}\left[-C_0^{\frac{2+\beta}{1-\beta}} \frac{1}{y} + \frac{2+\beta}{1-\beta} C_0^{\frac{1+2\beta}{1-\beta}} \ln y + \left(\frac{2+\beta}{1-\beta}\right)\left(\frac{1+2\beta}{1-\beta}\right) C_0^{\frac{3\beta}{1-\beta}} y + o(y)\right]_{s_1^{1-\beta}(\varepsilon_1)}^{s_2^{1-\beta}(\varepsilon_2)}. \tag{6.28}$$

In the same way, we can calculate I_4 and obtain

$$I_4 = \frac{1}{1-\beta}\left[-C_0^{\frac{2+\beta}{1-\beta}} \frac{1}{y} + \frac{2+\beta}{1-\beta} C_0^{\frac{1+2\beta}{1-\beta}} \ln y + \left(\frac{2+\beta}{1-\beta}\right)\left(\frac{1+2\beta}{1-\beta}\right) C_0^{\frac{3\beta}{1-\beta}} y + o(y)\right]_{s_3^{1-\beta}(\varepsilon_2)}^{1}. \tag{6.29}$$

$$\begin{aligned} I_2 + I_4 &= \frac{C_0^{\frac{2+\beta}{1-\beta}}}{1-\beta} \frac{1}{s_1^{1-\beta}(\varepsilon_1)} - \frac{C_0^{\frac{2+\beta}{1-\beta}}}{1-\beta} \left(\frac{1}{s_2^{1-\beta}(\varepsilon_2)} - \frac{1}{s_3^{1-\beta}(\varepsilon_2)}\right) \\ &\quad - C_0^{\frac{1+2\beta}{1-\beta}} \frac{2+\beta}{(1-\beta)^2} \ln s_1^{1-\beta}(\varepsilon_1) + C_0^{\frac{1+2\beta}{1-\beta}} \frac{2+\beta}{(1-\beta)^2} \left(\ln s_2^{1-\beta}(\varepsilon_2) - \ln s_3^{1-\beta}(\varepsilon_2)\right) + \text{finite terms} \end{aligned} \tag{6.30}$$

Given the results in last section, we have

$$\begin{aligned} \frac{1}{s_2^{1-\beta}(\varepsilon_2)} - \frac{1}{s_3^{1-\beta}(\varepsilon_2)} &= \frac{s_3^{1-\beta}(\varepsilon_2) - s_2^{1-\beta}(\varepsilon_2)}{s_2^{1-\beta}(\varepsilon_2) s_3^{1-\beta}(\varepsilon_2)} \\ &= \frac{\left(C_0^{\frac{1}{1-\beta}} + \frac{1}{1-\beta}\varepsilon_2 + o(\varepsilon_2)\right) - \left(C_0^{\frac{1}{1-\beta}} - \frac{1}{1-\beta}\varepsilon_2 + o(\varepsilon_2)\right)}{\left(C_0^{\frac{1}{1-\beta}} + \frac{1}{1-\beta}\varepsilon_2 + o(\varepsilon_2)\right)\left(C_0^{\frac{1}{1-\beta}} - \frac{1}{1-\beta}\varepsilon_2 + o(\varepsilon_2)\right)}, \end{aligned} \tag{6.31}$$

which is order of ε_2 as ε_2 approaches zero. Also note that

$$\begin{aligned}
\ln s_1^{1-\beta}(\varepsilon_1) &= (1-\beta)\ln\left[\left(\tfrac{\varepsilon_1}{C_0}\right)^{\frac{1}{\beta}}\left(1+\tfrac{1}{\beta C_0^{\frac{1}{\beta}-1}}\varepsilon_1^{\frac{1}{\beta}-1}+\tfrac{3-\beta}{2\beta^2 C_0^{\frac{2}{\beta}-2}}\varepsilon_1^{\frac{2}{\beta}-2}+o\left(\varepsilon_1^{\frac{2}{\beta}-2}\right)\right)\right]. \\
&= \tfrac{1-\beta}{\beta}\ln\varepsilon_1-\tfrac{1-\beta}{\beta}\ln C_0+(1-\beta)\ln\left[1+\tfrac{1}{\beta C_0^{\frac{1}{\beta}-1}}\varepsilon_1^{\frac{1}{\beta}-1}+\tfrac{3-\beta}{2\beta^2 C_0^{\frac{2}{\beta}-2}}\varepsilon_1^{\frac{2}{\beta}-2}+o\left(\varepsilon_1^{\frac{2}{\beta}-2}\right)\right]
\end{aligned} \tag{6.32}$$

For $\beta<1$, this implies that the order of $\ln s_1^{1-\beta}(\varepsilon_1)\to\infty$ when $\varepsilon_1\to 0$, is same as $\frac{1-\beta}{\beta}\ln\varepsilon_1$. Next we have

$$\ln s_2^{1-\beta}(\varepsilon_2)-\ln s_3^{1-\beta}(\varepsilon_2)=(1-\beta)\ln\frac{C_0^{\frac{1}{1-\beta}}-\frac{1}{1-\beta}\varepsilon_2+o(\varepsilon_2)}{C_0^{\frac{1}{1-\beta}}+\frac{1}{1-\beta}\varepsilon_2+o(\varepsilon_2)}, \tag{6.33}$$

which approaches zero as $\varepsilon_2\to 0$.

Given 6.19, we have

$$\frac{1}{s_1^{1-\beta}(\varepsilon_1)}=\frac{C_0^{\frac{1-\beta}{\beta}}}{\varepsilon_1^{\frac{1-\beta}{\beta}}}\cdot\left[1+\frac{1}{\beta C_0^{\frac{1}{\beta}-1}}\varepsilon_1^{\frac{1}{\beta}-1}+o\left(\varepsilon_1^{\frac{1}{\beta}-1}\right)\right]^{\frac{1-\beta}{\beta}} \tag{6.34}$$

$$\begin{aligned}
I_2+I_4 &= \tfrac{C_0^{\frac{2+\beta}{1-\beta}}}{1-\beta}\tfrac{1}{s_1^{1-\beta}(\varepsilon_1)}-C_0^{\frac{1+2\beta}{1-\beta}}\tfrac{2+\beta}{(1-\beta)^2}\ln s_1^{1-\beta}(\varepsilon_1)+\text{finite terms} \\
&= \tfrac{C_0^{\frac{2+\beta}{1-\beta}+\frac{1-\beta}{\beta}}}{(1-\beta)\varepsilon_1^{\frac{1-\beta}{\beta}}}-C_0^{\frac{1+2\beta}{1-\beta}}\tfrac{2+\beta}{\beta(1-\beta)}\ln\varepsilon_1+\text{finite terms.}
\end{aligned} \tag{6.35}$$

Therefore we have the following Lemma:

Lemma 6.1.2. *With the modification defined by* (6.1)*, the integral $I(\varepsilon_1,\varepsilon_2)$ given by* (6.24) *approaches ∞ as $(\varepsilon_1,\varepsilon_2)\to(0^+,0^+)$. For $\beta\in(0,1)$, the speed of "blow-up" of I is estimated by*

$$\lim_{\varepsilon_1\to 0^+} I(\varepsilon_1,\varepsilon_2)\varepsilon_1^{\frac{1-\beta}{\beta}}=\frac{C_0^{\frac{2+\beta}{1-\beta}+\frac{1-\beta}{\beta}}}{(1-\beta)} \tag{6.36}$$

for fixed ε_2 *and*

$$\lim_{\varepsilon_2 \to 0^+} I(\varepsilon_1, \varepsilon_2)\varepsilon_2 = \frac{2}{(1-\beta)\varepsilon_2}\left(1 + (\Delta - 1)C_0^{\frac{1}{1-\beta}} - \Delta C_0^{\frac{2}{1-\beta}}\right). \tag{6.37}$$

for fixed ε_1. *Especially for* $\beta = \frac{1}{2}$, *we have*

$$\lim_{\varepsilon_1 \to 0^+} I(\varepsilon_1, \varepsilon_2)\varepsilon_1 = 2C_0^6. \tag{6.38}$$

Let $C_1 = C_1(\varepsilon_1, \varepsilon_2)$ such that $\int_0^1 \rho(\varepsilon_1, \varepsilon_2, x)dx = 1$ and then we have

$$C_1(\varepsilon_1, \varepsilon_2) = \frac{1}{I(\varepsilon_1, \varepsilon_2)} \tag{6.39}$$

and

$$\rho(x, \infty, \varepsilon_1, \varepsilon_2) = \begin{cases} \frac{x^2}{I(\varepsilon_1,\varepsilon_2)\varepsilon_1^2}, & \text{for } 0 \le x \le s_1(\varepsilon_1), \\ \frac{x^2}{I(\varepsilon_1,\varepsilon_2)\sigma^2(x)}, & \text{for } s_1(\varepsilon_1) < x \le s_2(\varepsilon_2), \\ \frac{x^2}{I(\varepsilon_1,\varepsilon_2)\varepsilon_2^2}, & \text{for } s_2(\varepsilon_2) < x \le s_3(\varepsilon_2), \\ \frac{x^2}{I(\varepsilon_1,\varepsilon_2)\sigma^2(x)}, & \text{for } s_3(\varepsilon_2) < x \le 1. \end{cases} \tag{6.40}$$

Theorem 6.1.3. *Given the modification of* $\sigma^2(x)$*defined by* (6.1), *the dynamical behaviors of stationary solution* $\rho(x, \infty, \varepsilon_1, \varepsilon_2)$ *near* $x = 0^+$, $x = x^*$ *and* $x = 1$ *are determined by* ε_1, ε_2 *and their correlation when they both approach* 0^+.

1. $\rho(x, \infty, \varepsilon_1, \varepsilon_2)$ *is continuous in* ε_1 *and* ε_2 *at* $x = 0^+$ *and furthermore*

$$\lim_{x \to 0^+} \lim_{(\varepsilon_1,\varepsilon_2) \to (0^+,0^+)} \rho(x, \infty, \varepsilon_1, \varepsilon_2) = \lim_{(\varepsilon_1,\varepsilon_2) \to (0^+,0^+)} \lim_{x \to 0^+} \rho(x, \infty, \varepsilon_1, \varepsilon_2); \tag{6.41}$$

2. Assume that ε_2 *is a higher or the same order of* $\varepsilon_1^{\frac{1-\beta}{\beta}}$ *as* $(\varepsilon_1, \varepsilon_2) \to (0^+, 0^+)$. *Then the blow-up* *of* $\rho(x,\infty,\varepsilon_1,\varepsilon_2)$ *near* $x = x^*$ *is the same order of* $\frac{1}{\varepsilon}$ *and*

$$\rho(x,\infty,\varepsilon_1,\varepsilon_2) = \left[\frac{C_0^{\frac{2+\beta}{1-\beta}+\frac{1-\beta}{\beta}}\varepsilon_2}{(1-\beta)\varepsilon_1^{\frac{1-\beta}{\beta}}} + \frac{2}{(1-\beta)}\left(1+(\Delta-1)C_0^{\frac{1}{1-\beta}} - \Delta C_0^{\frac{2}{1-\beta}}\right) + \textit{finite terms}\right]^{-1}\frac{x^2}{\varepsilon_2}, \tag{6.42}$$

for $x \in (s_2(\varepsilon_2), s_3(\varepsilon_2))$. *If* $\varepsilon_1^{\frac{1-\beta}{\beta}}$ *is a higher order of* ε_2^2, *then* $\rho(x,\infty,\varepsilon_1,\varepsilon_2) \to 0$ *as* $(\varepsilon_1,\varepsilon_2) \to (0^+,0^+)$ *near* $x = x^*$. *Otherwise, the limit of p is undetermined.*

3. $\rho(x,\infty,\varepsilon_1,\varepsilon_2)$ *is smooth near* $x = x^*$ *and approaches zero as* $(\varepsilon_1,\varepsilon_2) \to (0^+,0^+)$.

Proof. In the interval $(0, s_1(\varepsilon_1))$, we have

$$\frac{x^2}{I(\varepsilon_1,\varepsilon_2)\varepsilon_1^2} \le \frac{s_1^2(\varepsilon_1)}{I(\varepsilon_1,\varepsilon_2)\varepsilon_1^2}. \tag{6.43}$$

From (6.19), this gives

$$\frac{x^2}{I(\varepsilon_1,\varepsilon_2)\varepsilon_1^2} \le \frac{\varepsilon_1^{\frac{2}{\beta}-2}}{I(\varepsilon_1,\varepsilon_2)C_0^{\frac{2}{\beta}}}\left[1+\frac{1}{\beta C_0^{\frac{1}{\beta}-1}}\varepsilon_1^{\frac{1}{\beta}-1} + \frac{3-\beta}{2\beta^2 C_0^{\frac{2}{\beta}-2}}\varepsilon_1^{\frac{2}{\beta}-2} + o\left(\varepsilon_1^{\frac{2}{\beta}-2}\right)\right]. \tag{6.44}$$

Since $0 < \beta < 1$, we have $\frac{2}{\beta} - 2 > 2$. Also note that $I(\varepsilon_1,\varepsilon_2) \to \infty$ as $(\varepsilon_1,\varepsilon_2) \to (0^+,0^+)$. Then we see the right hand side of the inequality goes to zero as $(\varepsilon_1,\varepsilon_2) \to (0^+,0^+)$ and thus

$$\lim_{x\to 0^+}\lim_{(\varepsilon_1,\varepsilon_2)\to(0^+,0^+)}\rho(x,\infty,\varepsilon_1,\varepsilon_2) = 0. \tag{6.45}$$

On the other hand, it's easy to see that $\frac{x^2}{I(\varepsilon_1,\varepsilon_2)\varepsilon_1^2} \to 0$ as $x \to 0^+$ and therefore

$$\lim_{(\varepsilon_1,\varepsilon_2)\to(0^+,0^+)}\lim_{x\to 0^+}\rho(x,\infty,\varepsilon_1,\varepsilon_2) = 0 \tag{6.46}$$

which gives the proof for the first part. For the second part, we have

$$\rho(x,\infty,\varepsilon_1,\varepsilon_2) = \frac{x^2}{I_1\varepsilon_2^2 + I_3\varepsilon_2^2 + (I_2+I_4)\varepsilon_2^2}. \tag{6.47}$$

So the blow-up of ρ at $x = x^*$ is determined by leading term of the denominator on the right hand side of the above equation. Based on the calculation given at the beginning of this section, the order of the leading term that we need to find should be the same as one of $\varepsilon^{\frac{3}{\beta}-2} \cdot \varepsilon_2^2$, $\frac{1}{\varepsilon_2} \cdot \varepsilon_2^2$, $\varepsilon^{\frac{1-\beta}{\beta}} \cdot \varepsilon_2^2$ and $\ln \varepsilon_1 \cdot \varepsilon_2^2$. Note that $\frac{3}{\beta} - 2 > 1$ and $\frac{1-\beta}{\beta} > 1$ since $0 < \beta < 1$. Then $\varepsilon^{\frac{3}{\beta}-2} \cdot \varepsilon_2^2$ and $\varepsilon^{\frac{1-\beta}{\beta}} \cdot \varepsilon_2^2$ are higher order terms of the $\frac{1}{\varepsilon_2} \cdot \varepsilon_2^2 = \varepsilon_2$ for sure. For the last term, we have

$$\ln \varepsilon_1 \cdot \varepsilon_2^2 = (\varepsilon_2 \ln \varepsilon_1)\varepsilon_2.$$

Since $x \ln x \to 0$ as $x \to 0^+$, we see that $\ln \varepsilon_1 \cdot \varepsilon_2^2 = o(\varepsilon_2)$. Therefore the leading term of the $I(\varepsilon_1, \varepsilon_2)\varepsilon_2^2$ is the same order of ε_2. The rest of proof follows by collecting the coefficients. Since there is no singularity of ρ in $(s_2(\varepsilon_2), s_3(\varepsilon_2))$, part three is straightforward by taking the limit on

$$\rho(x = 1, \infty, \varepsilon_1, \varepsilon_2) = \frac{1^2}{I(\varepsilon_1, \varepsilon_2)\sigma^2(1)}$$

as $(\varepsilon_1, \varepsilon_2) \to (0^+, 0^+)$. The proof is now complete. $\square$

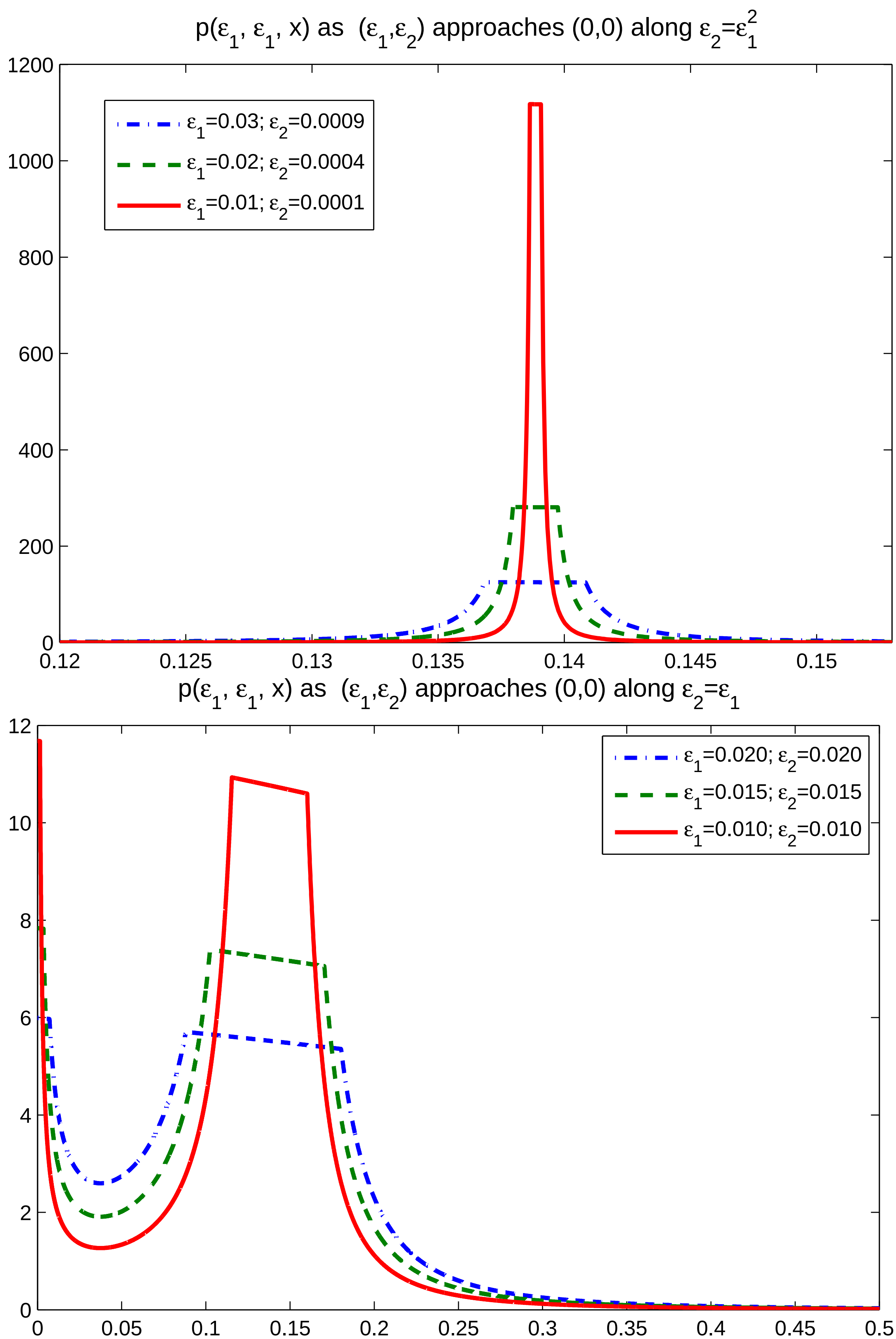

Figure 6.2: Stationary Solutions with Various Modifications. Top: Left End (near $x = 0$); Bottom: Right End (near $x = x^*$).

6.2 Extension Along the Tangent Line Near $x = 0$

With the simple and straightforward modification in section (6.1), as we have seen, the analysis at the first point involves an expansion of the cut-off point s_1 in terms of ε_1. Since the derivative of the non-modified function $\sigma^2(x)$ doesn't exist at zero, the type of modification appears too dramatic. In this section we try a new possible modification for the first point, where the horizon segement is replaced by the tangent line through the intersection point (see Figure 6.3).

In this case, for the first segement, we need to determine the tangent line of $\sigma^2(x)$ at point (s_1, ε_1) first and then find the intersection point of this line with the vertical line $x = 0$. Note that, at $x = s_1$, we have

$$\varepsilon_1(s_1) = (s_1 - C_0 s_1)^2, \tag{6.48}$$

which gives

$$\varepsilon_1'(s_1) = 2(s_1 - C_0 s_1^{\beta})(1 - C_0 \beta s^{\beta-1})_1. \tag{6.49}$$

Then the tangent line trough (s_1, ε_1) is given by

$$y - \varepsilon_1 = 2(s_1 - C_0 s_1^{\beta})(1 - C_0 \beta s_1^{\beta-1})(x - s_1), \tag{6.50}$$

which intersects with $x = 0$ at

$$y = \varepsilon_1 - 2s_1(s_1 - C_0 s_1^{\beta})(1 - C_0 \beta s_1^{\beta-1}). \tag{6.51}$$

In terms of parameter ε_1 we re-write the tangeng line as

$$\begin{aligned} y = \;& 2\left(s_1(\varepsilon_1) - C_0 s_1^{\beta}(\varepsilon_1)\right)\left(1 - C_0 \beta s_1^{\beta-1}(\varepsilon_1)\right) x \\ & + \varepsilon_1 - 2\left(s_1(\varepsilon_1) - C_0 s_1^{\beta}(\varepsilon_1)\right)\left(s_1(\varepsilon_1) - C_0 \beta s_1^{\beta}(\varepsilon_1)\right), \end{aligned} \tag{6.52}$$

where $s_1(\varepsilon_1)$ is given by Lemma (6.1.1). Then the modification of $\sigma^2(x)$ becomes

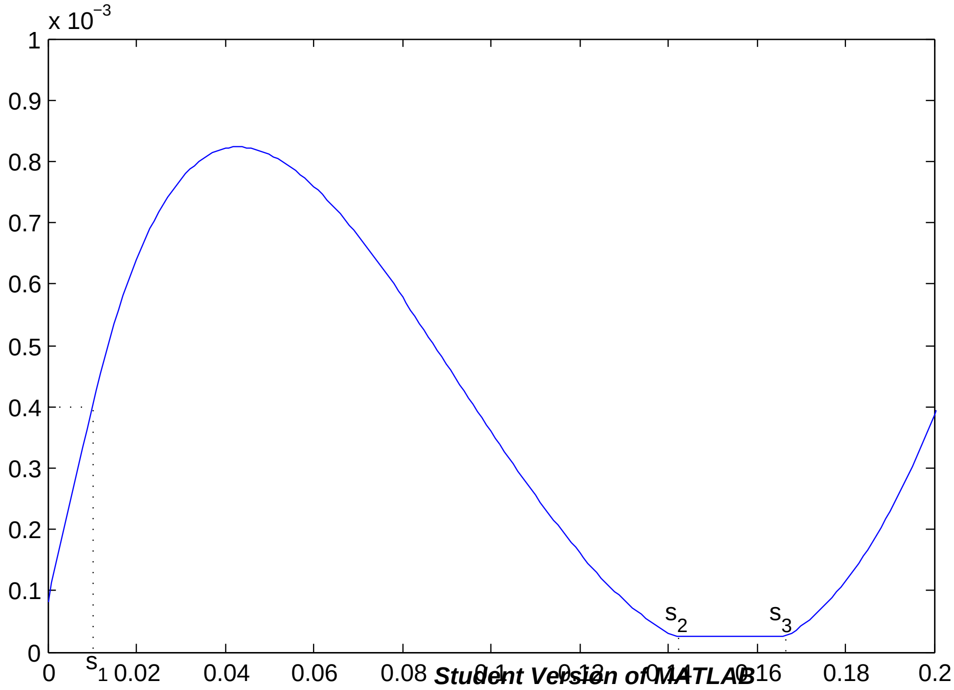

Figure 6.3: Second Modification of $\sigma^2(x)$.

$$
\sigma^2(\varepsilon_1,\varepsilon_2,x)
$$

$$
= \begin{cases} 2(s_1(\varepsilon_1)-C_0 s_1^{\beta}(\varepsilon_1))(1-C_0\beta s_1^{\beta-1}(\varepsilon_1))x+\varepsilon_1 \\ -2(s_1(\varepsilon_1)-C_0 s_1^{\beta}(\varepsilon_1))(s_1(\varepsilon_1)-C_0\beta s_1^{\beta}(\varepsilon_1)), \text{ for } 0\le x\le s_1(\varepsilon_1), \\ \sigma^2(x), \text{ for } s_1(\varepsilon_1)<x\le s_2(\varepsilon_2), \\ \varepsilon_2^2, \text{ for } s_2(\varepsilon_2)<x\le s_3(\varepsilon_2), \\ \sigma^2(x), \text{ for } s_3(\varepsilon_2)<x\le 1, \end{cases} \tag{6.53}
$$

where

$$
s_1(\varepsilon_1)=\left(\frac{\varepsilon_1}{C_0}\right)^{\frac{1}{\beta}}\left[1+\frac{1}{\beta C_0^{\frac{1}{\beta}-1}}\varepsilon_1^{\frac{1}{\beta}-1}+\frac{3-\beta}{2\beta^2 C_0^{\frac{2}{\beta}-2}}\varepsilon_1^{\frac{2}{\beta}-2}+o\left(\varepsilon_1^{\frac{2}{\beta}-2}\right)\right]. \tag{6.54}
$$

Based on this modification, $x=0$ is no longer a singular point for the original stochastic differential equation and the derived Fokker-Planck equation. In this case, the calculation it takes to determine the integral constant $C_1(\varepsilon_1,\varepsilon_2)$ such that $\int_0^1(x,\infty,\varepsilon_1,\varepsilon_2)dx=1$, is the same for I_2, I_3 and I_4. For

I_1, the integral is equal to the area of the trapezoidal enclosed by $x = 0$, $y = 0$, $x = s_1(\varepsilon_1)$ and the tangent line of $\sigma^2(x)$ at $(s_1(\varepsilon_1), \varepsilon_1)$. Thus we have

$$
\begin{aligned}
I_1(\varepsilon_1, \varepsilon_2) &= \tfrac{s_1(\varepsilon_1)}{2}\left[\varepsilon_1 + \left(\varepsilon_1 - 2(s_1(\varepsilon_1) - C_0 s_1^{\beta}(\varepsilon_1))(s_1(\varepsilon_1) - C_0\beta s_1^{\beta}(\varepsilon_1))\right)\right] \\
&= s_1(\varepsilon_1)\varepsilon_1 - s_1^{2\beta+1}(\varepsilon_1)(s_1^{1-\beta}(\varepsilon) - C_0)(s_1^{1-\beta}(\varepsilon) - C_0\beta).
\end{aligned}
\tag{6.55}
$$

Since $s_1(\varepsilon) \sim \varepsilon_1^{\frac{1}{\beta}}$, we have $s_1(\varepsilon)\varepsilon_1 \sim \varepsilon_1^{\frac{1+\beta}{\beta}}$ and $s_1^{2\beta+1}(\varepsilon_1) \sim \varepsilon_1^{\frac{2\beta+1}{\beta}}$. Therefore $I_1(\varepsilon_1, \varepsilon_2) \to 0$ as $(\varepsilon_1, \varepsilon_2) \to (0^+, 0^+)$ with the order of $\left(\frac{\varepsilon_1^{1+\beta}}{C_0}\right)^{\frac{1}{\beta}}$, given the estimation of s_1 in terms of ε_1.

Compared with the first modification, this modification shows an improvement at the $x = 0$. It appears that the stationary $\rho(x, \infty, \varepsilon_1, \varepsilon_2)$ is smooth at the first singular point and the integral I_1 can better estimate the corresponding integral based on the non-modified $\sigma^2(x)$. Since the modification at $x = x^*$ is same as the last section, the modification still only makes sense to the transformed FPE.

6.3 A Third Type of Modification

In this section, we look at a specific type of modification in the following form

$$
\sigma^2(x, \varepsilon) = \sigma^2(x) + h(x, \varepsilon) \tag{6.56}
$$

where $h(x, \varepsilon)$ is a smooth function such that (a) $h(x, \varepsilon) > 0$ for $x \geq 0$ and $\varepsilon > 0$; (b) $h(x, 0) = 0$ and $h(x, \varepsilon) \to 0$ as $\varepsilon \to 0$. In this case, we can see $\sigma(x, \varepsilon) \to 0$ asymptotically at the two singular points x^* and x_* as $\varepsilon \to 0$. For a better modification, we further assume that $h(x, \varepsilon)$ is small at x_* and x^* and decays fast as x moves away from x_* and x^*.

Actually it's easy to find a function that satisfies the above conditions. For example,

$$
h(x, \varepsilon) = \varepsilon\left[e^{-A^*(x-x^*)^2} + e^{-A_*(x-x_*)^2}\right], \tag{6.57}
$$

where $A^* > 0$ and $A_* > 0$ are large constants. We see that $e^{-A^*(x-x^*)^2}$ and $e^{-A_*(x-x_*)^2}$ are both equal to 1 but decay fast as x moves away from x^* and x_*. On the other hand, the asymptotic behavior of $h(x,\varepsilon) \to 0$ is controlled by ε (see Figure 5.5). Similarly we look at the following integral first

$$I(\varepsilon) = \int_0^1 \frac{x^2}{\sigma^2(x) + h(x,\varepsilon)} dx. \tag{6.58}$$

Note that

$$\lim_{x \to C_0^{\frac{1}{1-\beta}}} \frac{x - C_0 x^\beta}{x - C_0^{\frac{1}{1-\beta}}} = 1 - \beta. \tag{6.59}$$

With $C_0^{\frac{1}{1-\beta}} = x^*$, we have

$$\sigma^2(x) = (1-\beta)^2 (x - x^*)^2 + o((x-x^*)^2) \tag{6.60}$$

and thus

$$I(\varepsilon) = \frac{1}{(1-\beta)^2} \int_0^1 \frac{x^2}{(x-x^*)^2 + \left(\sqrt{\frac{h(x,\varepsilon) + o((x-x^*)^2)}{(1-\beta)^2}}\right)^2} dx. \tag{6.61}$$

Due to the singularity at $x = x^*$, $I(\varepsilon) \to \infty$ as $\varepsilon \to 0$. The blow-up speed of $I(\varepsilon)$ can be estimated as follows. Note that in a sufficiently small neighborhood $(x^* - \delta_0, x^* + \delta_0)$ of x^*, $(x-x^*)^2 << \varepsilon$ and

$$\sqrt{\frac{h(x,\varepsilon) + o((x-x^*)^2)}{(1-\beta)^2}} \approx \frac{\sqrt{\varepsilon}}{1-\beta}. \tag{6.62}$$

Then we have

$$\begin{aligned}
&\int_{x^*-\delta_0}^{x^*+\delta_0} \frac{x^2}{(x-x^*)^2 + \left(\sqrt{\frac{h(x,\varepsilon)+o((x-x^*)^2)}{(1-\beta)^2}}\right)^2} dx \\
&\approx \int_{x^*-\delta_0}^{x^*+\delta_0} \frac{x^2}{(x-x^*)^2 + \left(\frac{\sqrt{\varepsilon}}{1-\beta}\right)^2} dx \\
&= \left[\left(\frac{\sqrt{\varepsilon}}{1-\beta} - \frac{(1-\beta)(x^*)^2}{\sqrt{\varepsilon}} \right) \arctan\left(\frac{(1-\beta)(x^*-x)}{\sqrt{\varepsilon}} \right) + x^* \ln\left((x-x^*)^2 + \frac{\sqrt{\varepsilon}}{1-\beta} \right) + x \right]_{x^*-\delta_0}^{x^*+\delta_0} \\
&= 2 \left(\frac{(1-\beta)(x^*)^2}{\sqrt{\varepsilon}} - \frac{\sqrt{\varepsilon}}{1-\beta} \right) \arctan\left(\frac{(1-\beta)\delta_0}{\sqrt{\varepsilon}} \right) + 2\delta_0.
\end{aligned} \tag{6.63}$$

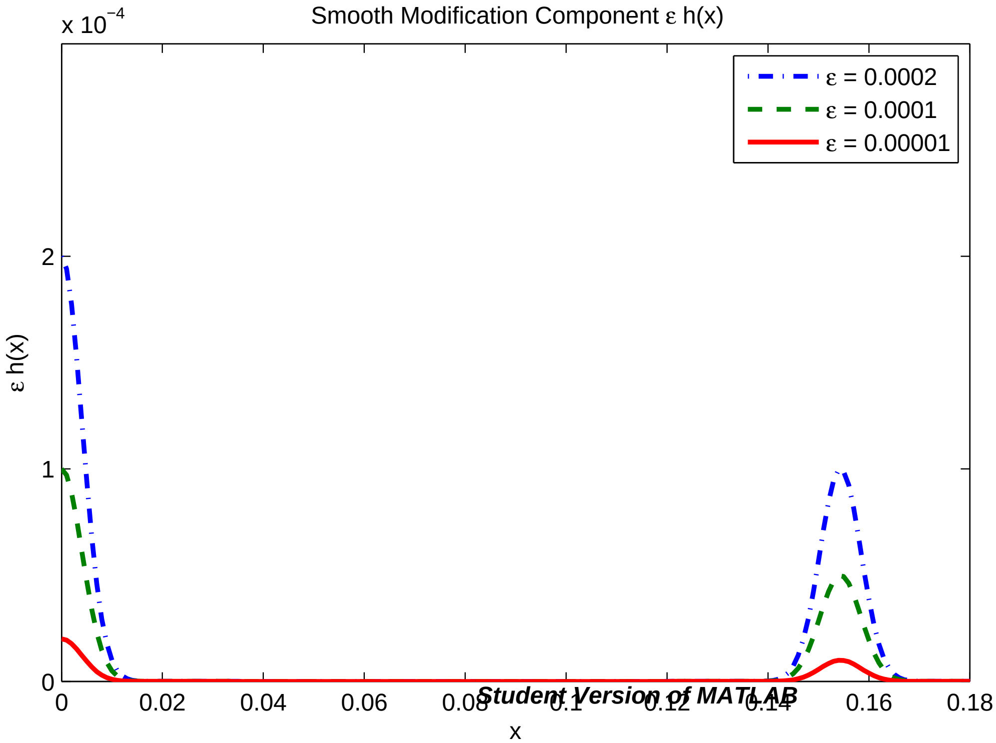

Figure 6.4: The Third Type of Modification with Different Parameters.

Therefore, we have

$$\begin{aligned} I(\varepsilon) &> \int_{x^*-\delta_0}^{x^*+\delta_0} \frac{x^2}{(x-x^*)^2+\left(\sqrt{\frac{h(x,\varepsilon)+o((x-x^*)^2)}{(1-\beta)^2}}\right)^2} dx \\ &> 2\left(\frac{(1-\beta)(x^*)^2}{\sqrt{\varepsilon}} - \frac{\sqrt{\varepsilon}}{1-\beta}\right) \arctan\left(\frac{(1-\beta)\delta_0}{\sqrt{\varepsilon}}\right) + 2\delta_0 \end{aligned} \tag{6.64}$$

and the estimation of the stationary solution as follows

$$\begin{aligned} \rho(x,\infty,\varepsilon) &= \frac{1}{I(\varepsilon)} \int_0^1 \frac{x^2}{\sigma^2(x)+h(x,\varepsilon)} dx \\ &< \left[2\left(\frac{(1-\beta)(x^*)^2}{\sqrt{\varepsilon}} - \frac{\sqrt{\varepsilon}}{1-\beta}\right) \arctan\left(\frac{(1-\beta)\delta_0}{\sqrt{\varepsilon}}\right) + 2\delta_0\right]^{-1} \int_0^1 \frac{x^2}{\sigma^2(x)+h(x,\varepsilon)} dx. \end{aligned} \tag{6.65}$$

For this type of modifications, the SDE and FPE are both meaningful for $\beta \geq \frac{1}{2}$ and is not subject to the transform since $h(x,\varepsilon)$ is differentiable for any $x \geq 0$. From Figure 6.5, we can see that the differences between the three modifications.

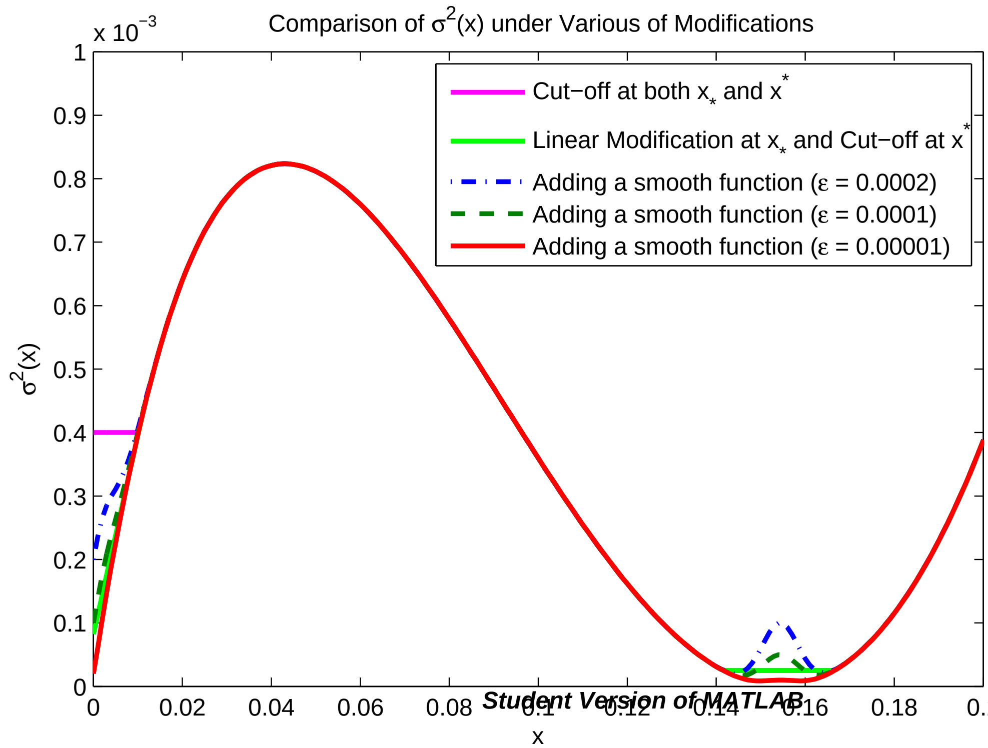

Figure 6.5: Comparison of the Three Modifications of $\sigma^2(x)$.

6.4 Time-Evolution of the Modified FPE with Numerical Experiments

In this section, we show some of the numerical results of the time-evolution for the modified FPE. Here we use the transformed equation

$$u_t(x,t,\varepsilon_1,\varepsilon_2) = \frac{1}{2}\sigma^2(x,\varepsilon_1,\varepsilon_2)u_{xx}(x,t,\varepsilon_1,\varepsilon_2), \tag{6.66}$$

equipped with initial condition

$$u(x,t=0) = \frac{1}{a\sqrt{\pi}}\frac{\sigma^2(x,\varepsilon_1,\varepsilon_2)}{x+\frac{1}{\Delta}}\exp-\frac{(x-\bar{\mu})^2}{a^2}, \tag{6.67}$$

where $\sigma(x,\varepsilon_1,\varepsilon_2)$ is the same as difined in the early section, a is a small positive number and $\bar{\mu}$ is a constant between 0 and 1. With the same setting of parameters α, β, r, F_0, Δ, σ_0, and a, the

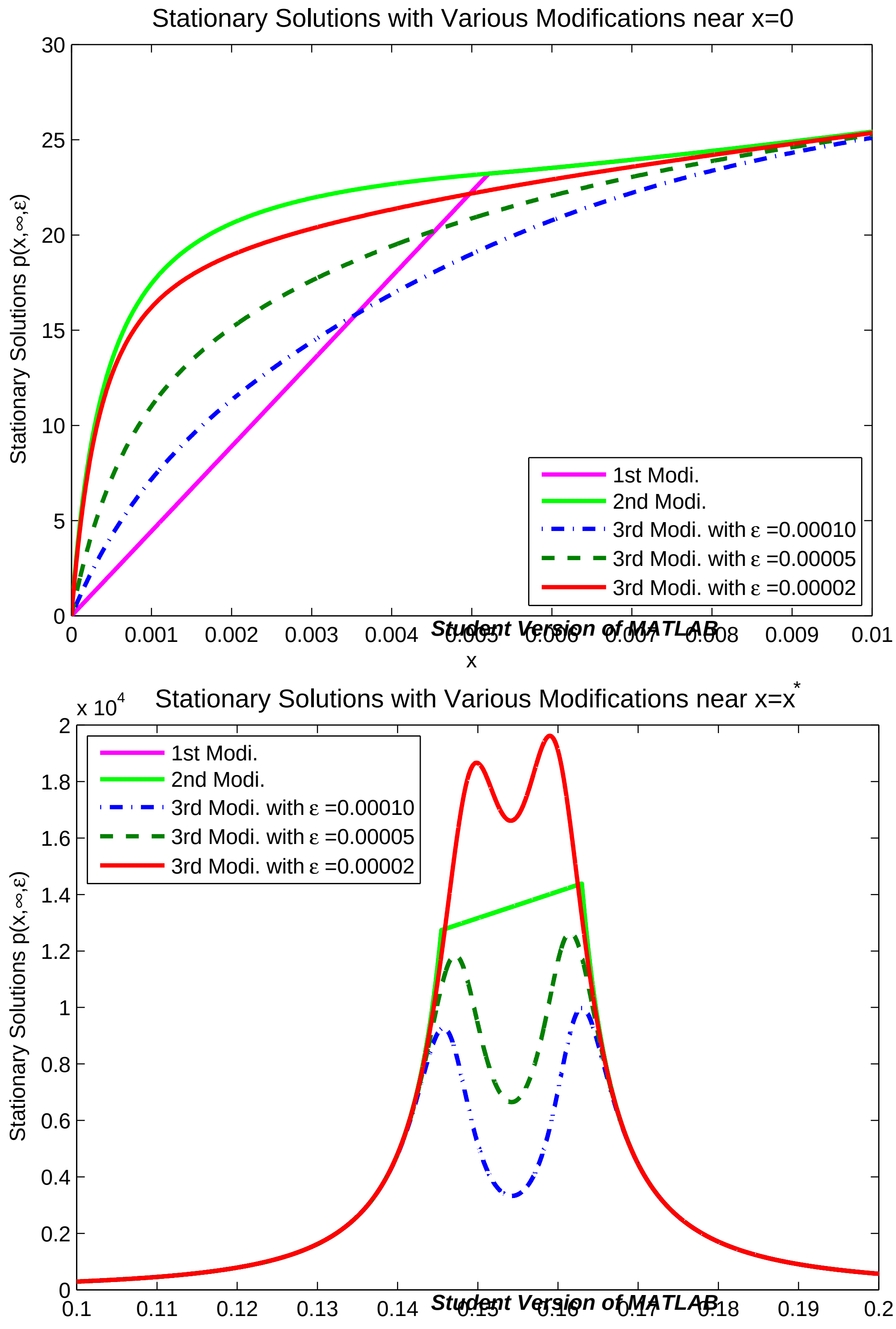

Figure 6.6: Comparison of Stationary Solutions with the Three Modifications.

evolution of the non-transformed equation is determined by ε_1 and ε_2. Some numerical experiments are done and the results are displayed in Figure 6.6, Figure 6.7 and Figure 6.8.

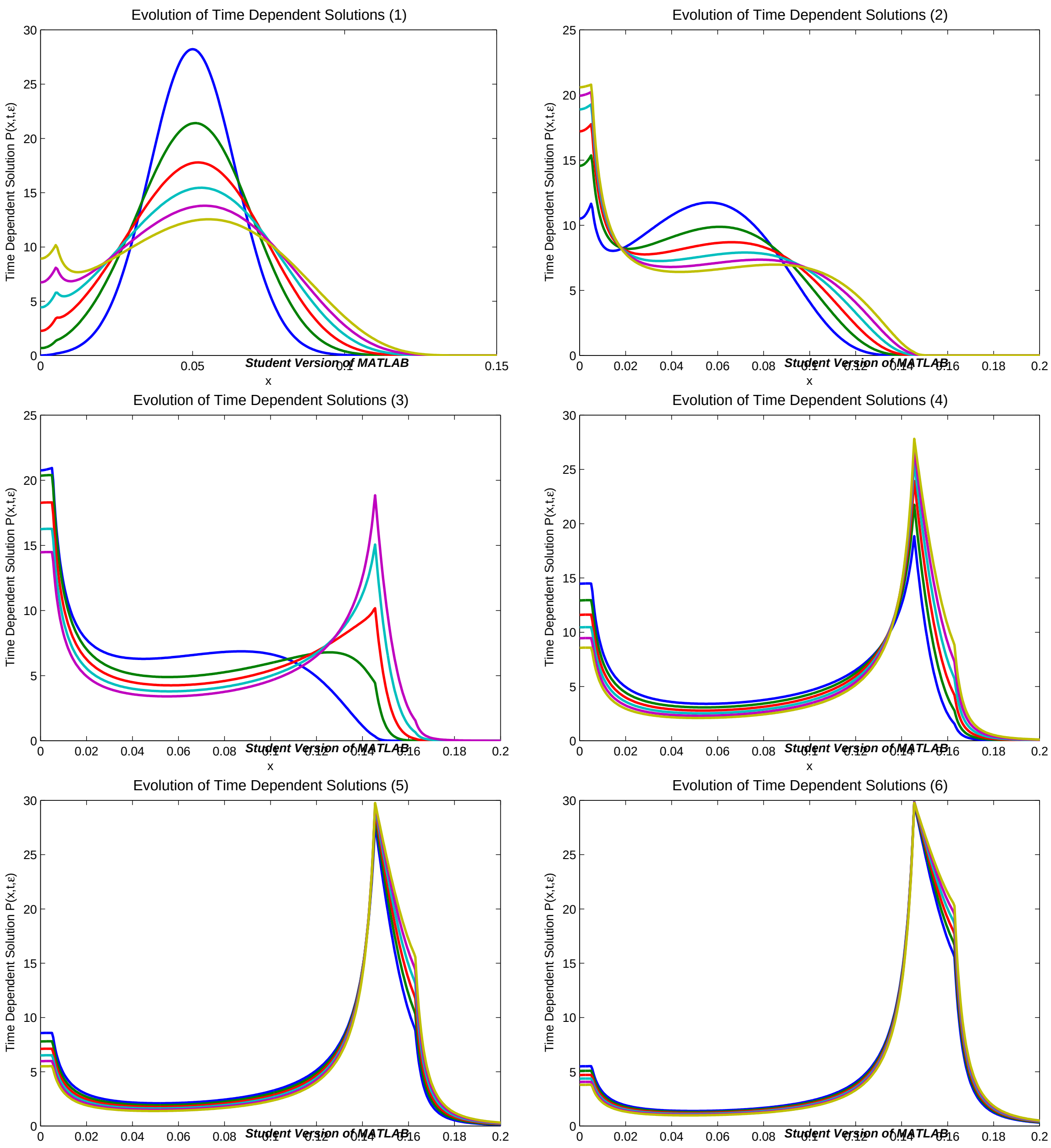

Figure 6.7: Time Evolution using the First Modification with Different Parameters. Upper Left: $\varepsilon_1 = 0.02$, $\varepsilon_2 = 0.02$; Upper Right: $\varepsilon_1 = 0.02$, $\varepsilon_2 = 0.01$; Middle Left: $\varepsilon_1 = 0.02$, $\varepsilon_2 = 0.005$; Middle Right: $\varepsilon_1 = 0.02$, $\varepsilon_2 = 0.0005$; Lower Left: $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.0005$; Lower Right: $\varepsilon_1 = 0.0075$, $\varepsilon_2 = 0.0005$;

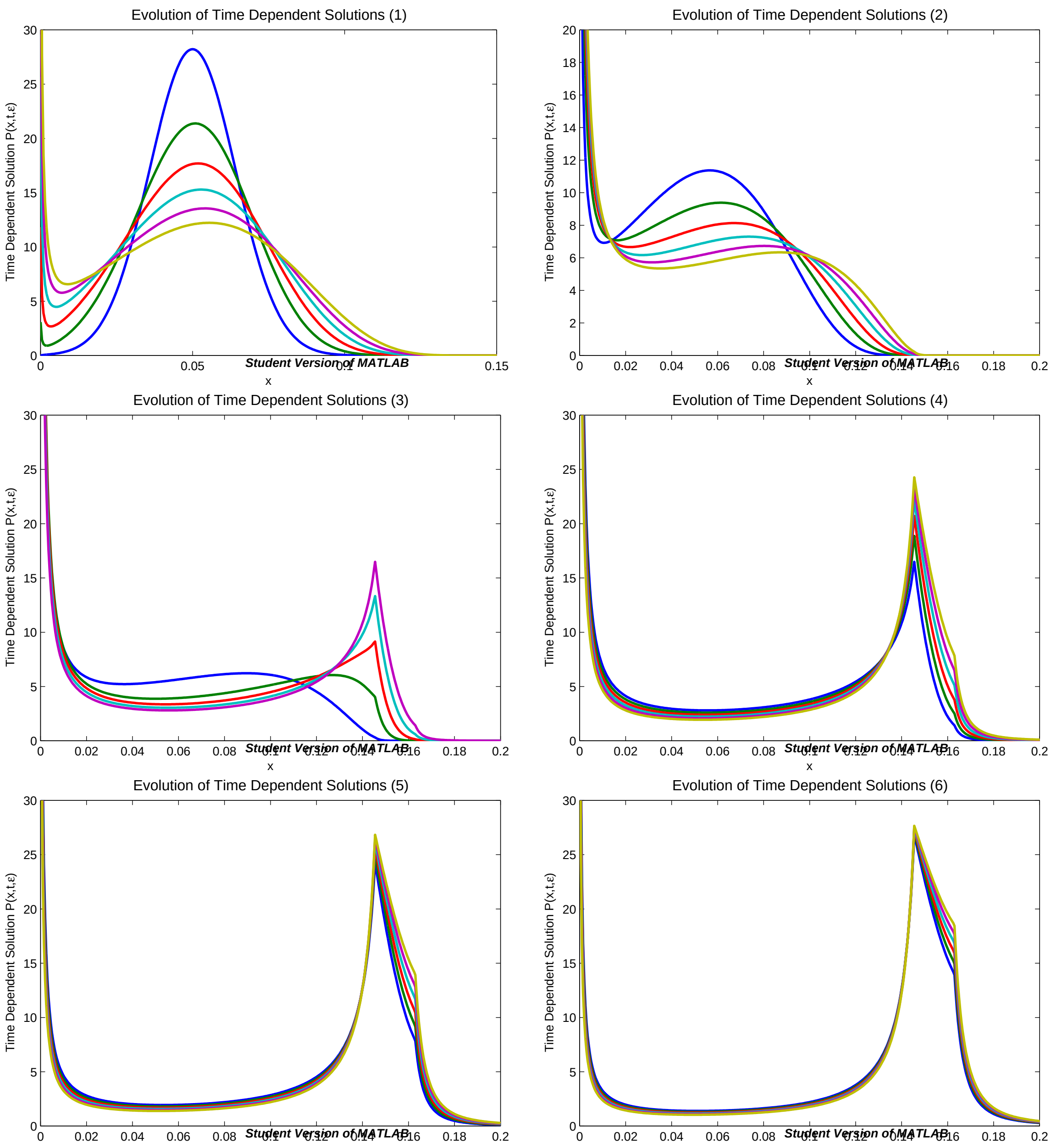

Figure 6.8: Time Evolution using the Second Modification with Different Parameters. Upper Left: $\varepsilon_1 = 0.02$, $\varepsilon_2 = 0.02$; Upper Right: $\varepsilon_1 = 0.02$, $\varepsilon_2 = 0.01$; Middle Left: $\varepsilon_1 = 0.02$, $\varepsilon_2 = 0.005$; Middle Right: $\varepsilon_1 = 0.02$, $\varepsilon_2 = 0.0005$; Lower Left: $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.0005$; Lower Right: $\varepsilon_1 = 0.0075$, $\varepsilon_2 = 0.0005$;

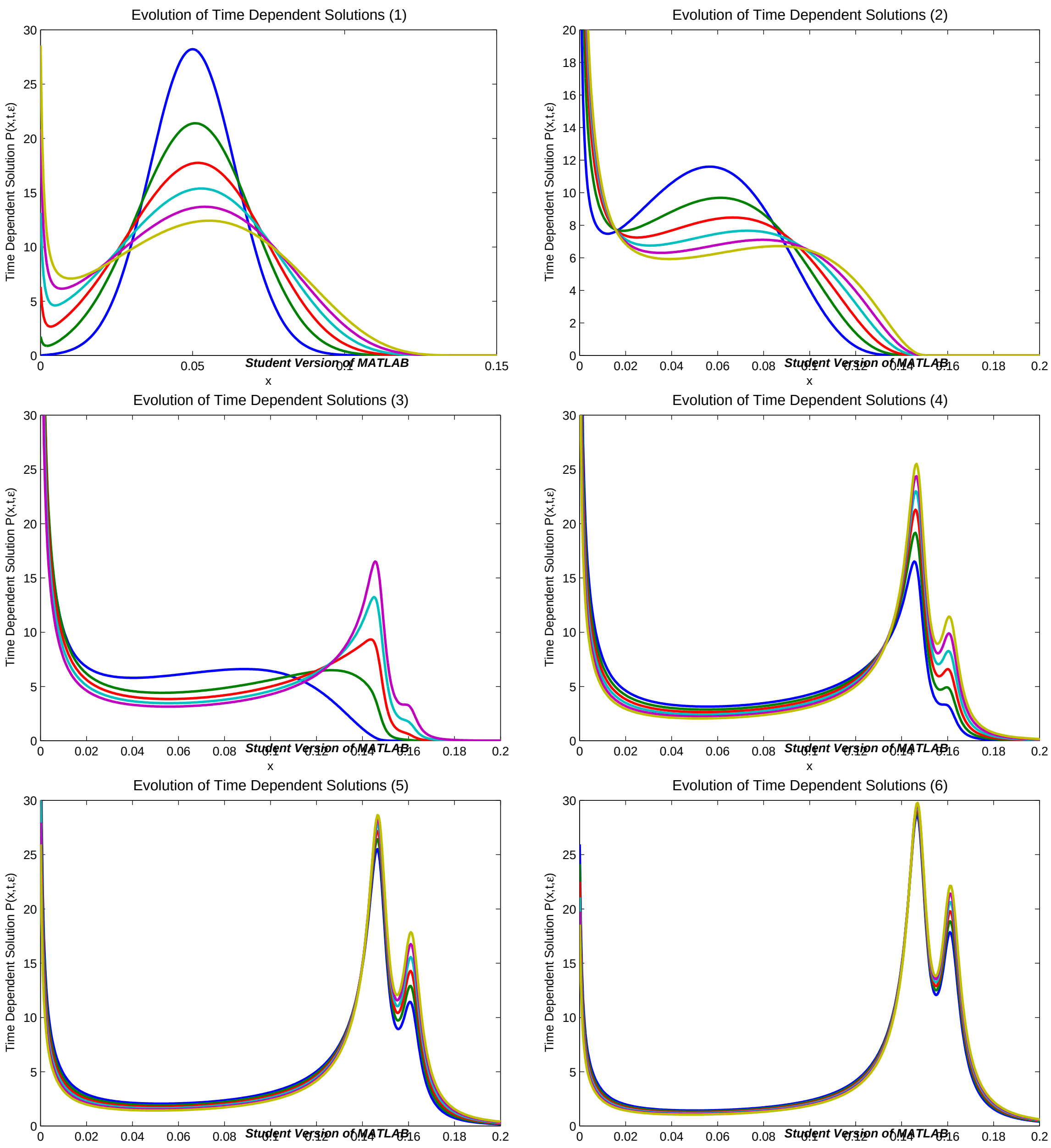

Figure 6.9: Time Evolution using the Third Modification with Different Parameters. Upper Left: $\varepsilon_1 = 0.02$, $\varepsilon_2 = 0.02$; Upper Right: $\varepsilon_1 = 0.02$, $\varepsilon_2 = 0.01$; Middle Left: $\varepsilon_1 = 0.02$, $\varepsilon_2 = 0.005$; Middle Right: $\varepsilon_1 = 0.02$, $\varepsilon_2 = 0.0005$; Lower Left: $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.0005$; Lower Right: $\varepsilon_1 = 0.0075$, $\varepsilon_2 = 0.0005$;

Chapter 7

Plan for Future Work

In this chapter, we show the study that we have initiated on the SDEs. The intention is to focus on the difference between the modified and the non-modified SDEs. Although the available result is only a little portion of the intended research, the numerical simulation results suggest that it points in a good direction, where we may obtain interesting and meaningful results if the study goes further in this direction. Given the time frame of this project, the rest of the research will be carried out in the future.

7.1 Difference between Modified and Non-modified SDEs

In this section, we compare the difference between the Modified and Non-modified SDEs. In general, we consider

$$dX_t = \mu(X_t)dt + \sigma^2(X_t)dW_t \tag{7.1}$$

and

$$dY_t = \mu(Y_t, \varepsilon)dt + \sigma^2(Y_t, \varepsilon)dW_t, \tag{7.2}$$

where $\mu(x, \varepsilon)$ and $\sigma(x, \varepsilon)$ are the same as defined in the previous sections. Since $\sigma^2(x, \varepsilon) \neq 0$ with the modification, multiplying (7.2) by $\frac{\sigma^2(X_t)}{\sigma^2(Y_t, \varepsilon)}$ with the given relationship between $\mu(x)$ and $\sigma(x)$, we have

$$\frac{\sigma^2(X_t)}{\sigma^2(Y_t,\varepsilon)}dY_t = \frac{\sigma^2(X_t)}{Y_t+\frac{1}{\Delta}}dt + \sigma^2(X_t)dW_t. \tag{7.3}$$

Subtracting (7.3) from (7.1), we have

$$dX_t - \frac{\sigma^2(X_t)}{\sigma^2(Y_t,\varepsilon)}dY_t = \left(\frac{\sigma^2(X_t)}{X_t+\frac{1}{\Delta}} - \frac{\sigma^2(X_t)}{Y_t+\frac{1}{\Delta}}\right)dt. \tag{7.4}$$

On the left hand side (LHS) of (7.4), we have

$$\begin{aligned} LHS &= dX_t - dY_t + dY_t - \tfrac{\sigma^2(X_t)}{\sigma^2(Y_t,\varepsilon)}dY_t \\ &= d(X_t - Y_t) + \left(\tfrac{\sigma^2(Y_t,\varepsilon)}{\sigma^2(Y_t,\varepsilon)} - \tfrac{\sigma^2(X_t)}{\sigma^2(Y_t,\varepsilon)}\right)dY_t. \end{aligned} \tag{7.5}$$

For the right hand side (RHS), we have

$$\begin{aligned} RHS &= \sigma^2(X_t)\left(\tfrac{1}{X_t+\delta} - \tfrac{1}{Y_t+\delta}\right)dt \\ &= \sigma^2(X_t)\tfrac{Y_t-X_t}{(X_t+\delta)(Y_t+\delta)}dt. \end{aligned} \tag{7.6}$$

Thus equation (7.4) becomes

$$d(X_t - Y_t) + \left(\frac{\sigma^2(Y_t,\varepsilon)}{\sigma^2(Y_t,\varepsilon)} - \frac{\sigma^2(X_t)}{\sigma^2(Y_t,\varepsilon)}\right)dY_t = -\sigma^2(X_t)\frac{X_t - Y_t}{(X_t+\delta)(Y_t+\delta)}dt, \tag{7.7}$$

which implies

$$d(X_t - Y_t) = -\frac{[X_t - Y_t]\sigma^2(X_t)}{(X_t+\delta)(Y_t+\delta)}dt - \frac{\sigma^2(Y_t,\varepsilon) - \sigma^2(X_t)}{\sigma^2(Y_t,\varepsilon)}dY_t. \tag{7.8}$$

Assuming $X_t \neq Y_t$, we have

$$\frac{d|X_t - Y_t|}{|X_t - Y_t|} = -\frac{\sigma^2(X_t)}{(X_t+\delta)(Y_t+\delta)}dt - \frac{\sigma^2(Y_t,\varepsilon) - \sigma^2(X_t)}{\sigma^2(Y_t,\varepsilon)|X_t - Y_t|}dY_t. \tag{7.9}$$

Using stochastic integrals, (6.9) can be written as

$$|X_t - Y_t| = e^{-\int_0^t \frac{\sigma^2(X_s)}{(X_s+\delta)(Y_s+\delta)}ds - \int_0^t \frac{\sigma^2(Y_s,\varepsilon)-\sigma^2(X_s)}{\sigma^2(Y_s,\varepsilon)|X_s-Y_s|}dY_s}. \tag{7.10}$$

Now we want to estimate the two integrals on the right hand side of (7.10). Assuming $X_t, Y_t > 0$ and noting that

$$(X_s+\delta)(Y_s+\delta) \le \frac{1}{2}\left[(X_t+\delta)^2 + (Y_t+\delta)^2\right], \tag{7.11}$$

then we have

$$-\int_0^t \frac{\sigma^2(X_s)}{(X_s+\delta)(Y_s+\delta)}ds \le -\int_0^t \frac{2\sigma^2(X_s)}{(X_t+\delta)^2+(Y_t+\delta)^2}ds. \tag{7.12}$$

From (7.2), we have

$$\begin{aligned} \int_0^t \frac{\sigma^2(Y_s,\varepsilon)-\sigma^2(X_s)}{\sigma^2(Y_s,\varepsilon)|X_s-Y_s|}dY_s &= \int_0^t \frac{\sigma^2(Y_s,\varepsilon)-\sigma^2(X_s)}{\sigma^2(Y_s,\varepsilon)|X_s-Y_s|}[\mu(Y_s,\varepsilon)ds + \sigma^2(Y_s,\varepsilon)dW_s] \\ &= \int_0^t \frac{\sigma^2(Y_s,\varepsilon)-\sigma^2(X_s)}{(X_s+\delta)|X_s-Y_s|}ds + \int_0^t \frac{\sigma^2(Y_s,\varepsilon)-\sigma^2(X_s)}{|X_s-Y_s|}dW_s \end{aligned} \tag{7.13}$$

7.2 Numerical Experiment and Simulation

For the given coefficients $\mu(x)$ and $\sigma(x)$, SDEs (7.1) and (7.2) can not be solved explicitly. In this section, we use a numerical method to provide approximated simulation of the two equations.

For SDE (7.1) with initial condition $X_0 = x$ and fixed time interval $[0,T]$, consider the partition

$$t_i = \frac{iT}{n}, i = 0, 1, \cdots, n,$$

where the length of each subinterval is $\delta_n = \frac{T}{n}$. Here we use the common approach, Euler's method that consists of the following recursive scheme

$$X^{(n)}(t_i) = X^{(n)}(t_{i-1}) + \mu(X^{(n)}(t_{i-1}))\delta_n + \sigma^2(X^{(n)}(t_{i-1}))(W_{t_i} - W_{t_{i-1}}), \tag{7.14}$$

where $i = 1, 2, \cdots, n$ and $X_0^{(n)} = x$. Inside the interval (t_{i-1}, t_i), the value of process $X^{(n)}$ is obtained by linear interpolation. The process $X^{(n)}$ is a function of the Brownian motion and we can measure the error of the scheme. Replacing X by $X^{(n)}$, we have

$$e_n = \sqrt{E\left[\left(X_T - X_t^{(n)}\right)^2\right]} \tag{7.15}$$

and it can be proved that e_n is the of the order $\delta^{\frac{1}{2}}$ or

$$e_n \leq c_n \delta^{\frac{1}{2}}$$

for $n \geq n_0$ and positive constant c_n.

In order to simulate a trajectory of the SDE solution of (7.1), it is sufficient to simulate the values of n independent random variables $\xi_1, \xi_2, \cdots, \xi_n$ with distribution $N(0,1)$ and then replace $W_{t_i} - W_{t_{i-1}}$ by $\sqrt{\delta_n}\xi_i$. Thus a possible path or realization of process X_t defined by (7.1) is the limit of $X^{(n)}(t_n)$ as follows

$$X^{(n)}(t_i) = X^{(n)}(t_0) + \lim_{n\to\infty}\left[\Sigma_{j=0}^{i-1}\mu(X^{(n)}(t_{j-1}))\delta_n + \Sigma_{j=0}^{i-1}\sigma^2(X^{(n)}(t_{j-1}))\sqrt{\delta_n}\xi_j\right]. \tag{7.16}$$

In application, one often needs to simulate expected value of certain stochastic process that is driven by X_t. For example, the payoff P of a financial instrument at the expiry date T is usually defined by

$$V \equiv E[P(X(T))] \tag{7.17}$$

and can be implemented by repeating the above simulation many times and then averaging over the outcomes. Usually each simulation generate one paths and the expected payoff at the expiry date T can be estimated by

$$V \approx \frac{1}{N}\Sigma_{k=1}^{N} P(X^{(n)}(T), k) \tag{7.18}$$

where k represents the k-th path, N is the total number of paths and n is the number of the partition nodes that is predetermined along each path.

Using the technique introduced above, we simulate both original SDE (7.1) and the modified SDE (7.2) separately and have observed some interesting results. The following figures are from the simulation using the parameter set $\beta = 0.6$, $\alpha = 0.1255$, $r = -0.7$, $\Delta = 2$, $\sigma_0 = 0.04$, $F_0 = 0.03$ (For the modified, the modification parameters are specified in the figures below).

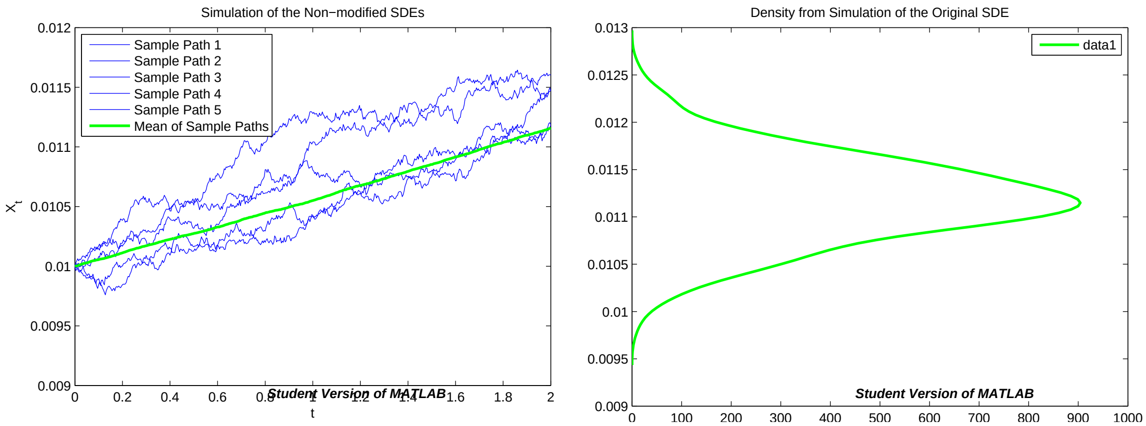

Figure 7.1: Simulation of the Non-modified SDEs.

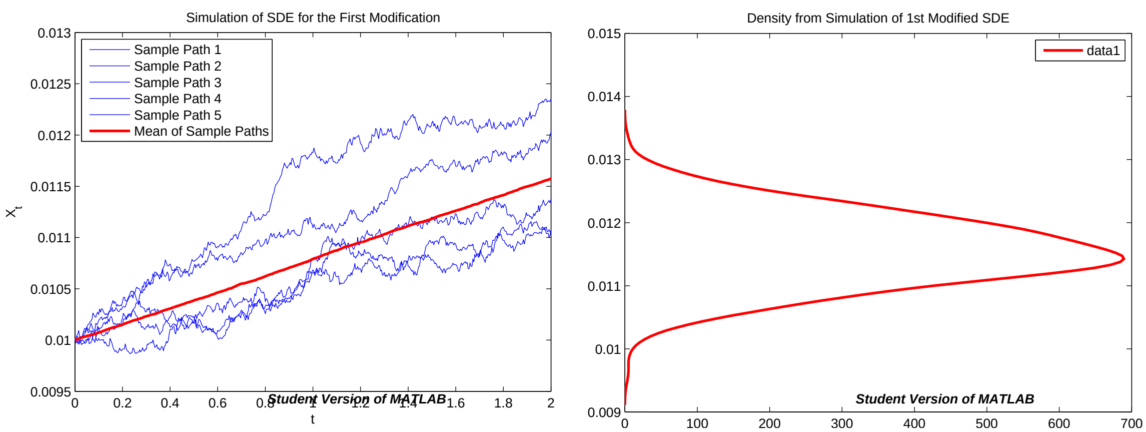

Figure 7.2: Simulation of the First Modification with $\varepsilon_1 = 0.02$, $\varepsilon_2 = 0.005$.

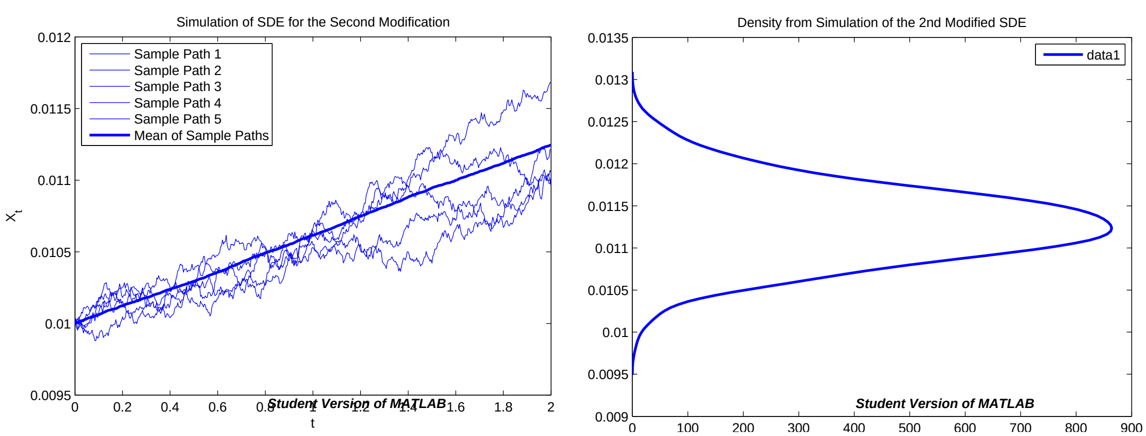

Figure 7.3: Simulation of the Second Modification with $\varepsilon_1 = 0.02$, $\varepsilon_2 = 0.005$.

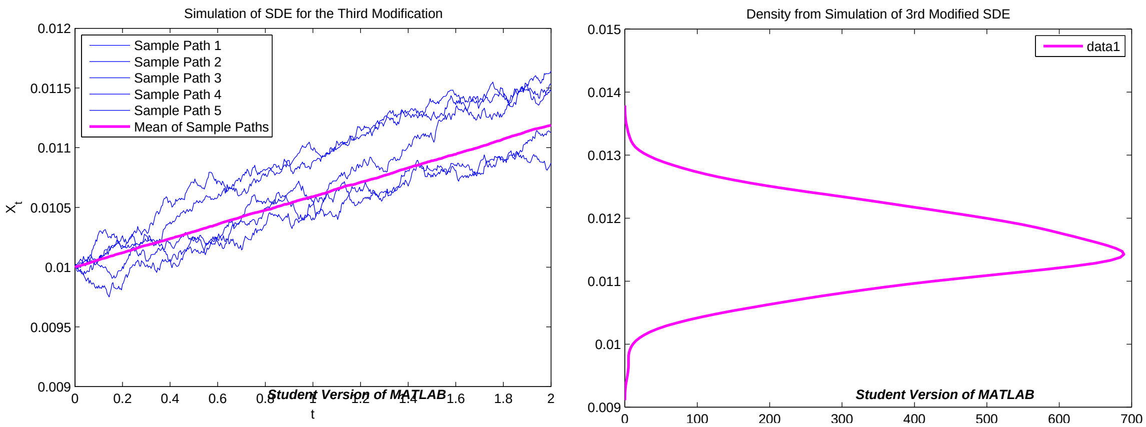

Figure 7.4: Simulation of the Third Modification with $\varepsilon_3 = 0.0001$.

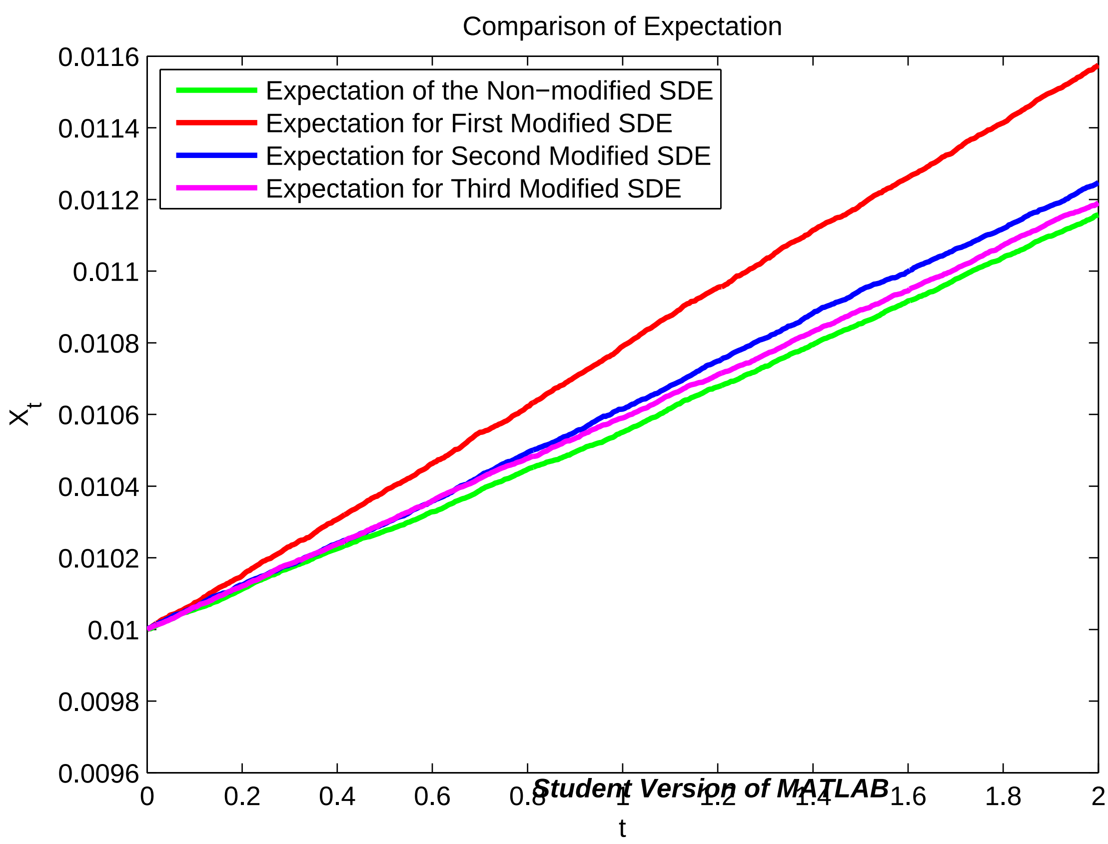

Figure 7.5: Expectations of the Processes defined by Non-Modified and Modified SDEs,using parameter set: $\beta = 0.6$, $\alpha = 0.1255$, $r = -0.7$, $\Delta = 2$, $\sigma_0 = 0.04$, $F_0 = 0.03$, $\varepsilon_2 = 0.02$, $\varepsilon_1 = 0.005$, $\varepsilon_3 = 0.0001$.

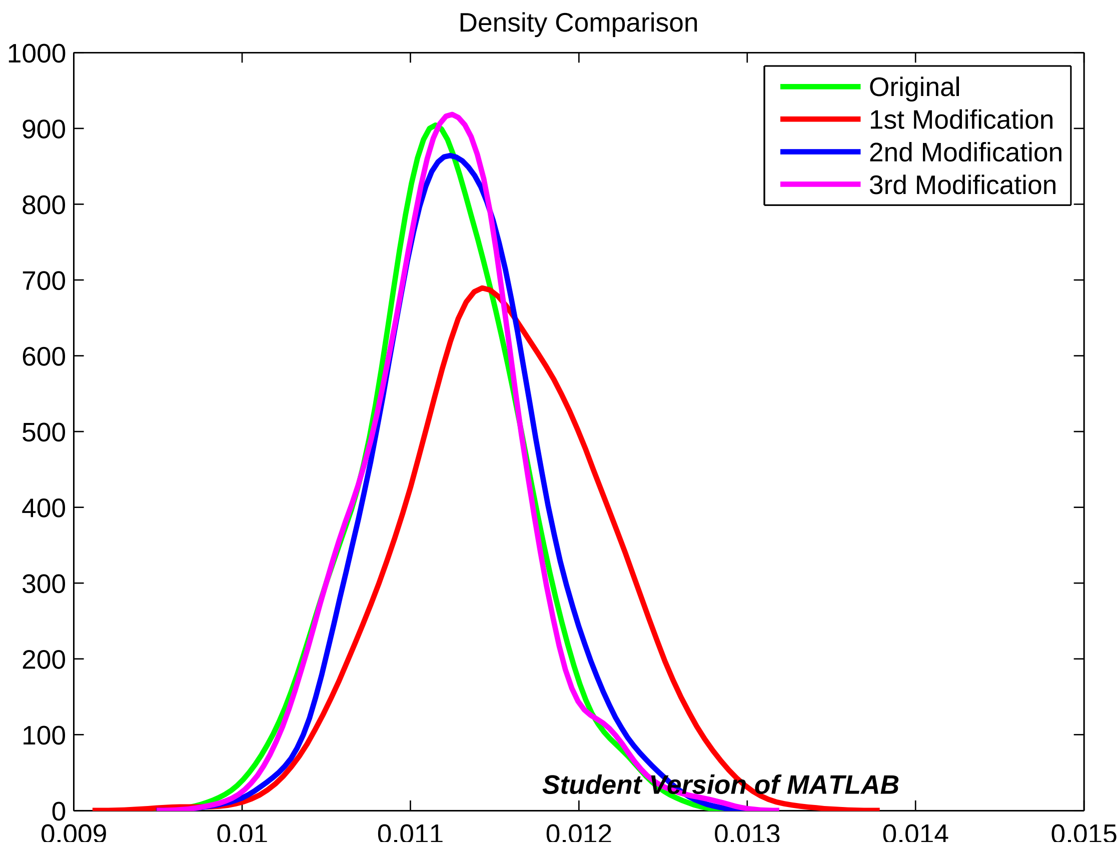

Figure 7.6: Expectations of the Processes defined by Non-Modified and Modified SDEs,using parameter set: using parameter set: $\beta = 0.6$, $\alpha = 0.1255$, $r = -0.7$, $\Delta = 2$, $\sigma_0 = 0.04$, $F_0 = 0.03$, $\varepsilon_2 = 0.02$, $\varepsilon_1 = 0.005$, $\varepsilon_3 = 0.0001$.

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Appendix A

An Example of Mat Lab Code for Numerical Solution to the Modified FPE using Backward Euler Method

```
%Set up parameters %
beta=0.6;
alpha=0.01255;
r=-0.7;
F0=0.03;
Delta=0.5;
Sigma0=0.005;
c0=F0^(1-beta)-(1-beta)*Sigma0/(alpha*r);
epsilon1=0.015;
epsilon2=0.0035;
epsilon31=0.0001;
epsilon32=0.00005;
epsilon33=0.00002;
```

```
epsilon3=epsilon33;

% Define mu, sigma and other related terms %
a1=0;
a2=(beta*c0)^(1/(1-beta));
a3=(c0)^(1/(1-beta));
a4=1;

fn=@(x)(c0*x.^(beta)-x).^2;
f1 = @(x) (c0*x.^(beta)-x).^2-epsilon1^2;
f2 = @(x) (c0*x.^(beta)-x).^2-epsilon2^2;
hx = @(x) exp(-30000*x.^(2))
+exp(-30000*(x-c0^(1/(1-beta))).^2)/2.*(x<=a4);
s1=bisection(f1,a1,a2);
s2=bisection(f2,a2,a3);
s3=bisection(f2,a3,a4);
sigma_md1 = @(x) (epsilon1^2).*(x<s1)
+(c0*x.^(beta)-x).^2.*(s1<=x).*(x<s2)
+epsilon2^2.*(s2<=x).*(x<s3)
+(c0*x.^(beta)-x).^2.*(s3<=x).*(x<=a4);
sigma_md2 = @(x)
(2*(s1-c0*s1^beta)*(1-c0*beta*s1^(beta-1))*(x-s1)
+epsilon1^2).*(x<s1)+(c0*x.^(beta)-x).^2.*(s1<=x).*(x<s2)\\
+epsilon2^2.*(s2<=x).*(x<s3)\\
+(c0*x.^(beta)-x).^2.*(s3<=x).*(x<=a4);
sigma_md31 = @(x) ((c0*x.^(beta)-x).^2+
epsilon31*hx(x)).*(x<=a4);
```

```
sigma_md32 = @(x) ((c0*x.^(beta)-x).^2
+epsilon32*hx(x)).*(x<=a4);
sigma_md33 = @(x) ((c0*x.^(beta)-x).^2
+epsilon33*hx(x)).*(x<=a4);

XL = 1; % A large number so that
mu / sig^2 (x) =0 for x > XL.
rho1= @(x) Delta*x.*(x+1/Delta)./sigma_md1(x);
rho2= @(x) Delta*x.*(x+1/Delta)./sigma_md2(x);
rho31= @(x) Delta*x.*(x+1/Delta)./sigma_md31(x);
rho32= @(x) Delta*x.*(x+1/Delta)./sigma_md32(x);
rho33= @(x) Delta*x.*(x+1/Delta)./sigma_md33(x);
%Plot rho %
xx=0:0.00001:0.3;
yy1=rho1(xx);
yy2=rho2(xx);
yy31=rho31(xx);
yy32=rho32(xx);
yy33=rho33(xx);

% Solving the FPE Numerically %
g = @(x) ((x+1/Delta).^2)./(sigma_md1(x).^2);
int_steps=1000;
I=trapzoidal(g,a1,a4,int_steps);
myA=1000/I;
myB=myA/Delta;
```

```
a=0.02;
pi=3.1415927;
initial_f = @(x) (1/a*pi^(1/2))*exp(-(x-0.05).^2/a^2)
.*((x-c0*x.^(beta)).^2./(x+1/Delta));
bound_L = @(t) 0;
bound_R = @(t) 0;
[u_md,x_md,t_md] = Implicit_Modified_Robin
(Delta,sigma_md1,initial_f,bound_L,bound_R,1000,2000);
p_time=u_md;

for j=1:length(t_md)
p_time(:,j)=u_md(:,j).*(x_md'+1./Delta)./sigma_md1(x_md');
        integral_temp = 0;
        for i=1:length(x_md)-1
        integral_temp =integral_temp
        + p_time(i,j)*(x_md(i+1)-x_md(i));
        end
p_time(:,j)=p_time(:,j)/integral_temp;
end
```

Appendix B

An Example of Mat Lab Code for Simulation of the SDEs using Euler–Maruyama Method

```
% Set up parameters %
beta=0.6;
alpha=0.1255;
r=-0.7;
F0=0.03;
Delta=2;
Sigma0=0.04;
c0=F0^(1-beta)-(1-beta)*Sigma0/(alpha*r);
epsilon1=0.02;
epsilon2=0.005;
epsilon3=0.0001;

M=500;
```

```
N=500;
t0=0;
t1=2;
dt=(t1-t0)/N;
t=t0:dt:t1;

%%%%%%%%%%%%% Original Sigma %%%%%%%%%%%
sigma_orig = @(x) (c0*x.^(beta)-x).^2;
mu_orig=@(x) Delta.*sigma_orig(x)./(Delta.*x+1);

%%%%%%%%%%%% Modified Sigma %%%%%%%%%%%%
a1=0;
a2=(beta*c0)^(1/(1-beta));
a3=(c0)^(1/(1-beta));
a4=1;

fn=@(x)(c0*x.^(beta)-x).^2;
f1 = @(x) (c0*x.^(beta)-x).^2-epsilon1^2;
f2 = @(x) (c0*x.^(beta)-x).^2-epsilon2^2;
hx = @(x) exp(-30000*x.^(2))
+exp(-30000*(x-c0^(1/(1-beta))).^2)/2.*(x<=a4);
s1=bisection(f1,a1,a2);
s2=bisection(f2,a2,a3);
s3=bisection(f2,a3,a4);

sigma_1 = @(x) (epsilon1^2).*(x<s1)
+(c0*x.^(beta)-x).^2.*(s1<=x).*(x<s2)
```

```
+epsilon2^2.*(s2<=x).*(x<s3)
+(c0*x.^(beta)-x).^2.*(s3<=x).*(x<=a4);
mu_1=@(x) Delta.*sigma_1(x)./(Delta.*x+1);
sigma_2 = @(x) (2*(s1-c0*s1^beta)
*(1-c0*beta*s1^(beta-1))
*(x-s1)+epsilon1^2).*(x<s1)
+(c0*x.^(beta)-x).^2.*(s1<=x).
*(x<s2)+epsilon2^2.*(s2<=x).*(x<s3)
+(c0*x.^(beta)-x).^2.*(s3<=x).*(x<=a4);
mu_2=@(x) Delta.*sigma_2(x)./(Delta.*x+1);
sigma_3 = @(x) ((c0*x.^(beta)-x).^2
+epsilon3*hx(x)).*(x<=a4);
mu_3=@(x) Delta.*sigma_3(x)./(Delta.*x+1);

%%%%%%%%%%%% Simulation of
the non-modified SDE%%%%%%%%%%%%%%
X = zeros(1,length(t));
ALLX_Orig=zeros(M,length(t));
X(1)=0.01;
for i=1:M
dW = sqrt(dt)*random('Normal',0,1,1,length(t));
for j=2:N+1
    X(j)=X(1)+
    sum(dt*mu_orig(X(1:j-1)))+sum(dW(1:j-1)
    .*sigma_orig(X(1:j-1)));
end
ALLX_Orig(i,:)=X;
```

```
end
EXPX_Orig=mean(ALLX_Orig);
h=figure(1);
plot(t,ALLX_Orig(10,:)); hold on
plot(t,ALLX_Orig(70,:)); hold on
plot(t,ALLX_Orig(100,:)); hold on
plot(t,ALLX_Orig(250,:)); hold on
plot(t,ALLX_Orig(400,:)); hold on
plot(t,EXPX_Orig,'g','LineWidth',2); hold off
xlabel('t');
ylabel('X_t');
legend('Sample Path 1','Sample Path 2','Sample Path 3',
'Sample Path 4','Sample Path 5','Mean of Sample Paths');
legend('location', 'NorthWest');
title('Simulation of the
Non-modified SDEs','FontSize',10);
ti = get(gca,'TightInset');
set(gca,'Position',[ti(1) ti(2)
1-ti(3)-ti(1) 1-ti(4)-ti(2)]);
set(gca,'units','centimeters')
pos = get(gca,'Position');
ti = get(gca,'TightInset');
set(gcf, 'PaperUnits','centimeters');
set(gcf, 'PaperSize', [pos(3)+ti(1)
+ti(3) pos(4)+ti(2)+ti(4)]);
set(gcf, 'PaperPositionMode', 'manual');
set(gcf, 'PaperPosition',[0 0 pos(3)
```

```
+ti(1)+ti(3) pos(4)+ti(2)+ti(4)]);
saveas(h,'Simulation_Non-mod.pdf');

%%%%%%%%%%%%%% Simulation of
the 1st-modified SDE%%%%%%%%%%%%%%%%
X = zeros(1,length(t));
ALLX_1=zeros(M,length(t));
X(1)=0.01;
for i=1:M
dW = sqrt(dt)*random('Normal',0,1,1,length(t));
for j=2:N+1
    X(j)=X(1)+ sum(dt*mu_1(X(1:j-1)))
    +sum(dW(1:j-1).*sigma_1(X(1:j-1)));
end
ALLX_1(i,:)=X;
end
EXPX_1=mean(ALLX_1);
```